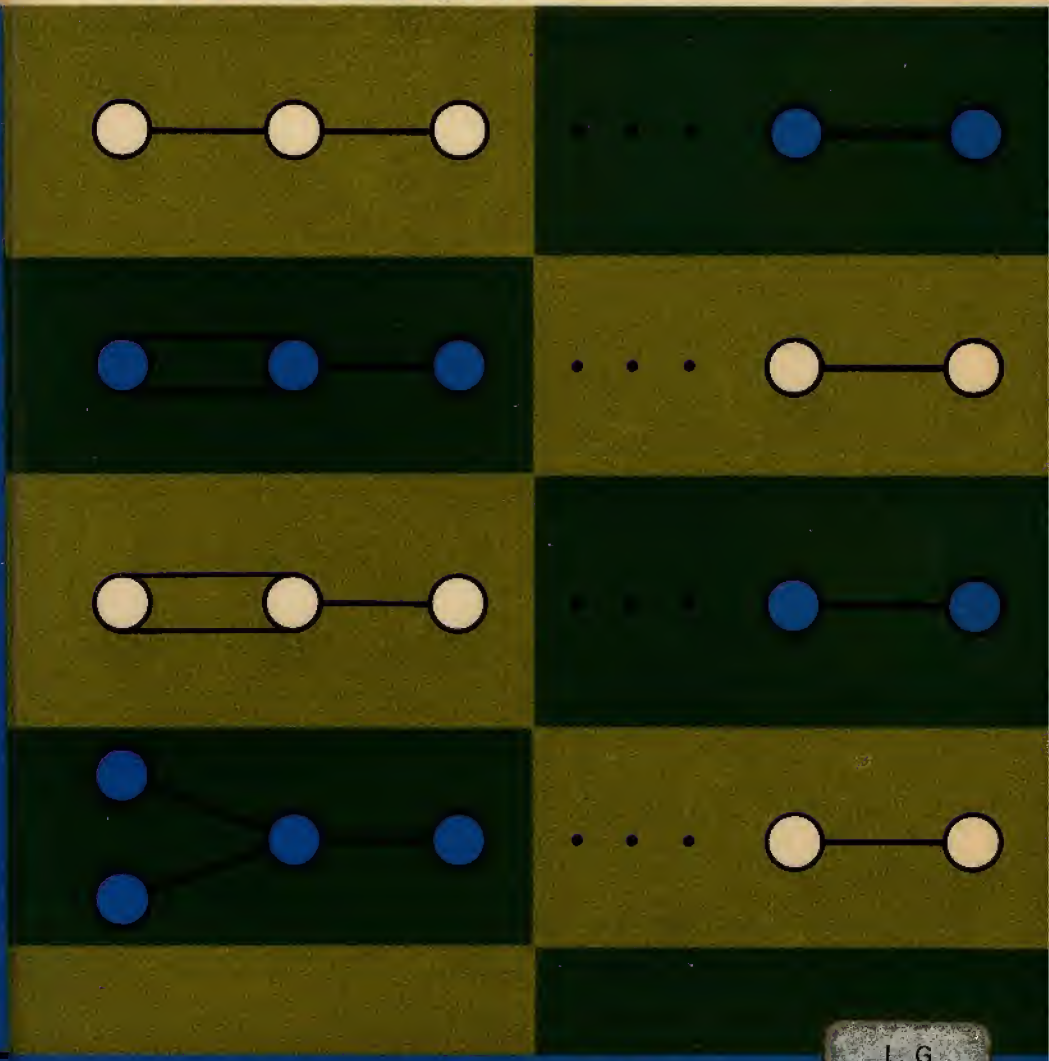
GROUP THEORY AND ELEMENTARY PARTICLES

P. A. ROWLATT



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GROUP THEORY AND ELEMENTARY PARTICLES

Group Theory and Elementary Particles

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PREFACE

The purpose of this book is to provide physicists with an introduction to the mathematics of the simple Lie algebras which underlie the symmetries of the strongly interacting particles. It provides a set of instructions on the use of simple Lie algebras rather than a mathematical description of them. The theory is developed from first principles but with few proofs of theorems given. Instead, theorems are stated and reference made to the whereabouts of the proof. There are numerous examples of the use of the theorems.

In the introduction a brief description of the background physics is given. In Chapter 1 the concepts of sets, groups, algebras, continuous groups, etc., are defined in order to give a clear understanding of the nature of a simple Lie algebra. In Chapter 2 the simple Lie algebras are classified and their structure described. In Chapter 3 the general properties of their representations are described together with the methods of calculating them. Particles and their spin states are taken to be in one-to-one correspondence with the basic vectors of the representations. It follows that the numbers of particles or states to a multiplet depend on the dimensions of the representations. In Chapters 4, 5 and 6 the methods of calculating matrix elements explicitly are described.

Figures in bold print within brackets (e.g. [**9**]) refer to the corresponding reference in the bibliography at the end of the book.

I should like to thank Dr R. F. Streater for reading the manuscript and for his many valuable comments. I am grateful to Mrs A. Blackman, Miss T. Morrisey and Miss F. Williams for typing the manuscript. Finally my husband, Dr Charles Rowlatt, has been a source of continual encouragement and occasional helpful criticism.

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INTRODUCTION

One of the greatest puzzles in recent years has been the proliferation of the so-called elementary particles. Only a short time ago there were believed to exist some thirty subatomic particles, namely, the photon, three or four leptons and their antiparticles, seven mesons, eight baryons and their antiparticles. Now another sixty or seventy particles have been discovered, all of them like the original baryons and mesons involved in strong interactions.

It now seems that all these particles are different states of the same entity. This idea first appeared when it was realised that the proton and neutron are in fact identical except for the difference in their charge and the small mass difference. The mass difference is attributable to their different behaviour with respect to electromagnetic interactions. This symmetry is expressed by calling the proton and neutron the two possible states of a ' nucleon ', an entity possessing a property known as ' isotopic spin ' in analogy with the spin of an electron. Just as an electron has two spin states, a left-handed spin state and a right-handed spin state, so the nucleon has two isotopic spin states, the proton with a positive charge and the neutron with no charge. In general, something is said to have a symmetry if there is an operation (such as reflection or rotation) which leaves it unaltered. A system has isotopic spin symmetry if it is invariant or unaltered by the application of three operators which are related by the rules of the simplest Lie algebra, that corresponding to the Lie group SU(2). All strongly interacting particles fit into isotopic spin multiplets. The π mesons and the Σ particles form triplets, the K mesons and Ξ particles form more doublets like the nucleons, while the Λ particle is a singlet.

It is possible to look on a doublet (perhaps the nucleons) as being basic. Higher multiplets can be formed by combining two or more of these basic entities. A combination of two produces an object with four states which is ' reducible ', separating into an SU(2) triplet like the Σ particles and an SU(2) singlet such as the Λ particle.

The next step was to extend the isotopic spin symmetry. The original baryons—the nucleon, the Σ particles, the Λ, and the Ξ particles—are in many ways very similar. By postulating a sym-

metry corresponding to the next higher Lie algebra, that corresponding to the group SU(3), Y. Ne'eman [1] and Murray Gell-Mann independently found these original baryons to form a supermultiplet. This symmetry is violated by medium strong interactions in a way which produces the mass differences between the baryon isotopic spin multiplets in the same way as the symmetry within the multiplets is violated by the electromagnetic interactions, to produce mass differences within the multiplets. This is the theory called by M. Gell-Mann [2] the Eightfold Way. In this case a basic entity can be postulated having three states and called (by M. Gell-Mann) a ' quark '. Particles are constructed by combining quarks and antiquarks. Mesons are formed by one quark and one antiquark, making nine possible states. These split into an SU(3) octet and an SU(3) singlet. The octet is the octet of mesons formed by the π mesons, the K mesons and the η meson. Baryons are formed by the combination of three quarks. The twenty-seven resulting states separate into a decimet, two octets and a singlet. The original eight baryons form one of the octets. When this theory was originated there were nine known particles fitting into the decimet. The properties of the final particle, the omega-minus, were predicted by the theory, and its discovery in February 1964 strikingly verified the SU(3) symmetry.

It is now established that in addition to their spin angular momentum (e.g. electron spin), which we have been taking for granted, and which corresponds to SU(2), the particles have the above SU(3) symmetry. In August 1964 it was postulated by F. Gursey and L. A. Radicati [3] and independently by B. Sakita [4] that these two symmetries might be combined in a larger symmetry corresponding to the group SU(6). In this theory the quarks have six components, three with left-handed and three with right-handed spin angular momentum. The combination of a quark with an antiquark to form meson states results in thirty-six possible states. On analysis these form a thirty-fivefold multiplet and a singlet. The thirty-fivefold multiplet breaks down with respect to SU(2) and SU(3) into an octet of spinless particles, an octet of particles with three spin states and a singlet with three spin states. The octet of spinless particles is the original octet of π mesons, K mesons and the η meson. More recently, another set of mesons has been discovered. They are heavier than these, have three spin states and form into an octet and a singlet as predicted by the theory.

Combining three quarks to form baryons leads to a set of two hundred and sixteen states which splits into a number of SU(6) multiplets. The one of interest is a fifty-sixfold multiplet consisting of eight particles with two spin states and ten particles with four spin states. This exactly fits the baryons discovered.

The major fault of the SU(6) symmetry is that it is non-relativistic, i.e. the equations are valid only for low velocities of the particles. In February 1965 Professor A. Salam, R. Delbourgo and J. Strathdee [5] extended the symmetry again. In this case the relevant group is SU(6, 6) which is related to SU(12).† The twelve quark states consist of three quarks each having four spin states. This theory is relativistic in the same way as Dirac's four-component theory of the electron and equation of motion is relativistic. Equations of motion imposed on the SU(6, 6) theory reduce the symmetry for any particular value of the momentum to SU(6) as above. The group SU(6, 6) contains as a subgroup the group O(3, 1),‡ which is the real Lorentz group. Invariance under the real Lorentz group expresses relativistic invariance and a theory invariant under SU(6, 6) would be automatically invariant under its subgroups.

In this book the mathematics of the simple Lie algebras underlying the above-mentioned Lie groups is outlined.

† SU(12) is the Lie group of 12×12 matrices U satisfying

$$U^{\dagger}GU = G; \qquad \det U = 1,$$

where G is known as the metric tensor and is the 12×12 unit matrix. On the other hand, SU(6, 6) is the Lie group of 12×12 matrices satisfying the same equation but with G the 12×12 diagonal matrix whose non-zero elements include $+1$ six times and -1 six times.

‡ O(3, 1) is related to O(4) the orthogonal group in four dimensions but has -1 three times and $+1$ once on the diagonal of the metric tensor G.

CHAPTER 1

Basic Concepts

This chapter is concerned with defining the concepts which we shall be using later on in the book. The collected details of notation can be found in Appendix 2.

1.1 Sets [6]

A set is any kind of collection of entities of any sort. It may be the set of all Frenchmen, the set of all integers, the set of all functions of two variables, or the set of all Frenchmen and functions of two variables. It may be the set of two apples and three oranges chosen from a fruit bowl, or the set with the integers 2, 5 and 121 for members.

The members of the set are said to belong to the set, and it is customary to use the symbol $\in$ to denote ' belongs to ', thus

Oxford University $\in$ the set of all English Universities

or

$2 \in$ the set of all integers.

On the other hand 0·5 does not belong to the set of all integers, and this is written

$$0{\cdot}5 \notin \mathscr{N}$$

where $\mathscr{N}$ represents the set of all integers.

Subset

If $\mathscr{A}$ and $\mathscr{B}$ are sets such that every member of $\mathscr{B}$ is also a member of $\mathscr{A}$, the set $\mathscr{B}$ is called a subset of the set $\mathscr{A}$ or $\mathscr{B}$ ' is included in ' $\mathscr{A}$. This is generally written

$$\mathscr{B} \subseteq \mathscr{A}.$$

If $\mathscr{B} \subseteq \mathscr{A}$ but $\mathscr{B} \neq \mathscr{A}$, that is, there are members of $\mathscr{A}$ which are not members of $\mathscr{B}$, we can say $\mathscr{B}$ ' is a proper subset ' of $\mathscr{A}$ and write

$$\mathscr{B} \subset \mathscr{A}.$$

Some examples follow:

The set of all women $\subset$ the set of all humans.
The set of all integers $\subset$ the set of all real numbers.
The set of vectors spanning a three-dimensional space $\subset$ the set of vectors spanning a four-dimensional space.
The set 2, 6, 7 $\not\subset$ the set 2, 7, 9, 11.
The set 2, 6, 7 $\subseteq$ the set 2, 6, 7.

Linear space

A linear space is a non-empty set $\mathscr{A}$ in which the following operations performed on elements of $\mathscr{A}$ produce entities which are also elements of $\mathscr{A}$.

(i) Multiplication by a complex number, i.e. if $X \in \mathscr{A}$ then $\alpha X \in \mathscr{A}$, where α is a complex number.
(ii) Addition of elements, i.e. if $X, Y \in \mathscr{A}$ then

$$X+Y=Z \in \mathscr{A}.$$

These operations must obey the usual laws of arithmetic, for instance, the addition must be associative, i.e.

$$(X+Y)+Z=X+(Y+Z)$$

must hold for all elements X, Y and $Z \in \mathscr{A}$.

Vector space

A vector space is a linear space in which there is also defined an operation of taking a scalar product, i.e. corresponding to every pair of elements of $\mathscr{A}$ there exists a complex number written (X, Y) called the scalar product of the elements X and Y.

A number of elements $X_1, X_2, \ldots, X_n$ of a vector space $\mathscr{A}$ are said to be linearly dependent if complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ can be found such that

$$\alpha_1 X_1+\alpha_2 X_2+ \ \ldots \ +\alpha_n X_n=0, \tag{1}$$

where all the α's are not zero. If there is no relation of this kind the elements are said to be linearly independent. The largest number of linearly independent elements of the set $\mathscr{A}$ is called the dimension of the vector space $\mathscr{A}$, assumed finite in this book.

Subspace

A subspace of a vector space is a subset of $\mathscr{A}$ such that the operations of multiplication of an element of the subset by a complex number

or addition of two elements form elements of the subset, i.e. if $\mathscr{N}$ is a subset of $\mathscr{A}$, $\mathscr{N} \subset \mathscr{A}$ then

$$\alpha X \in \mathscr{N} \quad \text{for} \quad X \in \mathscr{N}$$

and

$$X+Y \in \mathscr{N} \quad \text{for} \quad X, Y \in \mathscr{N}.$$

A subspace $\mathscr{N}$ is a proper subspace of $\mathscr{A}$ if there are members of $\mathscr{A}$ which are not members of $\mathscr{N}$.

Consider the set of all the vectors in three dimensions having complex amplitudes. Since multiplication of a vector by a complex number or addition of two vectors will result in another vector of the space, and a scalar product can be defined (in the usual way) the set is a vector space. The largest number of linearly independent members is three, being the three basic vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ chosen such that the scalar product of any pair is zero. C^3 is said to be spanned by the vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. The set can be written as

$$C^3 \equiv \{\mathbf{X}=\alpha_1 \mathbf{e}_1+\alpha_2 \mathbf{e}_2+\alpha_3 \mathbf{e}_3\} \tag{2}$$

or the set of all vectors of the form

$$\mathbf{X}=\alpha_1 \mathbf{e}_1+\alpha_2 \mathbf{e}_2+\alpha_3 \mathbf{e}_3,$$

where α_1, α_2 and α_3 are complex numbers.

The subset

$$C^2 \equiv \{\mathbf{X}=\alpha_1 \mathbf{e}_1+\alpha_2 \mathbf{e}_2\} \tag{3}$$

is a subspace of the vector space C^3. It is the subspace spanned by the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$.

Binary operation

A binary operation is an operation by which two members of a set form a third member of the set, thus a simple example of a binary operation in a linear space is the operation of addition referred to above. It is because of the binary operations that a set is restricted as to the type of members it can have if it is to be a linear space, vector space, etc. The set of all Frenchmen can never form a linear space, etc., as there can be no definition of a binary operation by which two elements combine to form a third element of the set. Entities which do form linear spaces, vector spaces, etc., may be numbers, vectors, functions, operators, matrices, and so on. To define a linear space or a vector space, it is not enough merely to name the elements of the set $\mathscr{A}$; one must also specify what

operations on the set are to be taken as multiplication by a complex number, addition of elements, and so on.

Algebra

An algebra is a linear space in which there is also defined a binary operation of multiplication, by which the product of any two elements of the set is also a member of the set, i.e. if ' o ' represents the binary operation of multiplication:

$$X \circ Y = Z \in \mathscr{A}, \qquad X, Y \in \mathscr{A}.$$

For example, the set of all complex numbers forms an algebra with respect to multiplication of complex numbers as it is usually defined. Consider the set C^3 of all the vectors in three dimensions with complex amplitudes. The operation of taking a scalar product will not form the set into an algebra since the scalar product of two vectors is not a member of the set. On the other hand, taking the vector product of two of the vectors does result in another vector in the space and therefore is a binary operation with which the set forms an algebra.

Subalgebra

A subalgebra is a subspace of the algebra such that any two members of the subspace when combined by the binary operation of multiplication form another member of the subspace.

A proper subalgebra is analogous to a proper subset. It is a subalgebra of an algebra which has some members which are not members of the subalgebra. Thus a proper subalgebra has fewer members than the algebra.

Group

A group is a set with a binary operation o which satisfies the following three axioms:

(i) The binary operation o is ' associative ', that is

$$X \circ (Y \circ Z) = (X \circ Y) \circ Z, \tag{4}$$

where X, Y and $Z \in$ the group $\mathscr{G}$.

(ii) There exists in the group $\mathscr{G}$ an identity element E such that

$$X \circ E = X, \qquad X \in \mathscr{G}. \tag{5}$$

(iii) Each element of the group has an inverse element with respect to the operation o, thus for every $X \in \mathscr{G}$ there exists an element X^{-1} such that

$$X \circ X^{-1} = E. \tag{6}$$

The symbol ' $^{-1}$ ' is called the ' unary operation symbol '.

The most familiar example of a group is the set of all integers, positive, negative and zero, and the binary operation of addition. The unary operation is then interpreted as the operation of negating a number and the identity element E is interpreted as zero. The three axioms can then be written

$$\left.\begin{array}{ll} \text{(i)} & X+(Y+Z)=(X+Y)+Z \\ \text{(ii)} & X+0=X \\ \text{(iii)} & X+(-X)=0 \end{array}\right\} \tag{7}$$

The fact that these three equations are true when X, Y and Z are positive or negative integers or zero is sufficient to prove that this set forms a group with respect to the binary operation of addition.

On considering the binary operation of multiplication we see that, although the set of all integers forms an algebra with respect to this operation, it does not form a group. The three axioms may be written

$$\left.\begin{array}{ll} \text{(i)} & X.(Y.Z)=(X.Y).Z \\ \text{(ii)} & X.1=X \\ \text{(iii)} & X.X^{-1}=1 \end{array}\right\} \tag{8}$$

where X, Y and Z are integers. However, X^{-1} is not a member of the set of integers and therefore this algebra is not a group.

On the other hand, the set of all positive rational numbers (that is, numbers of the form X/Y, where X and Y are positive integers, zero being excluded) does form a group with respect to the binary operation of multiplication. The unary operation is the operation of taking the inverse and the identity element is unity. The three axioms look the same as in equation (8) but in this case X, Y and Z are positive rational numbers, so X^{-1} is a member of the set.

The set of all vectors in three dimensions is not a group with respect to the binary operation of taking a vector product since, if $\mathbf{X}$ is a vector and $\wedge$ the vector product symbol, there is no identity element $\mathbf{E}$ in the set such that

$$\mathbf{X} \wedge \mathbf{E} = \mathbf{X}. \tag{9}$$

Subgroup

A subgroup is a subset of the members of a group which, by themselves, form a group. Thus any two members of the subgroup, when combined by the binary operation of the group, form a member of the subgroup. In addition the subgroup must contain the identity element and the inverse element of every member in the subgroup. If the group axiom (i) holds for the group, it will hold for the subgroup.

An example of a subgroup is the group of all positive and negative integers divisible by three, and zero together with the binary operation of addition. This is a subgroup of the group considered earlier and is a ' proper ' subgroup since the group contains members (i.e. integers not divisible by three) which are not contained in the subgroup.

Transformations and operators

An operator is an entity which when applied to an appropriate vector or function, etc., transforms it into another vector or function.

Examples of operators are: the identity operator which will leave anything it is applied to unchanged; multiplicative operators which are real numbers or complex numbers and multiply whatever they are applied to by their own value; differential operators, for instance $\partial/\partial x$, when applied to a function of x, transform it into a different function of x. Many other simple examples will occur to the reader.

A more complex example is a rotation operator. When applied to a vector this will rotate the vector about a specified axis and through a specified angle. Thus in three dimensions the rotation operator $P_z(\theta)$ applied to any vector transforms it into a different vector by rotating its projection on the plane perpendicular to the axis, in this case the z-axis, through an angle θ in that plane (see Fig. 1.1). This type of operator is known as a ' continuous ' operator because it depends on the quantity θ which can vary continuously. Any such transformation can be built up out of an infinite number of infinitesimal transformations $P_z(\mathrm{d}\theta)$.

Consider the question of whether these rotation operators form a group. Taking the binary operation to be repeated application of

the operators, we see that they do combine to give another element of the same kind since

$$P_z(\theta)P_z(\phi)\mathbf{v} = P_z(\theta+\phi)\mathbf{v}, \tag{10}$$

where $\mathbf{v}$ is a vector in the space, and $P_z(\theta+\phi)$ is a member of the set. Similarly

$$P_z(\theta)P_y(\phi)\mathbf{v} = P_\omega(\psi)\mathbf{v}, \tag{11}$$

where ω is the vector in the yz-plane about which the rotation $P_\omega(\psi)$ takes place and ψ is the angle of the rotation. Both the direction of ω and the magnitude of ψ can be found in terms of θ and ϕ. The operator $P_\omega(\psi)$ is a member of the set of all rotation operators in three dimensions.

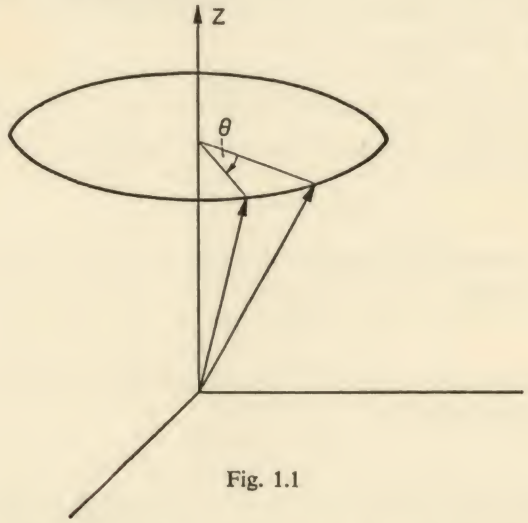


Fig. 1.1

The binary operation of repeated application is associative. The set contains the identity operator E which leaves vectors unchanged

$$E\mathbf{v} = \mathbf{v} \tag{12}$$

$$P_z(\theta)E\mathbf{v} = P_z(\theta)\mathbf{v} \tag{13}$$

because we choose that it should. For each operator $P_z(\theta)$ there is an inverse operator $P_z(-\theta)$ such that

$$P_z(\theta)P_z(-\theta)\mathbf{v} = E\mathbf{v} = \mathbf{v}. \tag{14}$$

It follows that the set of rotation operators in three dimensions forms a group with respect to the binary operation of repeated application.

1.2 Lie algebras

Lie group

The rotation group in three dimensions is an example of a Lie group. A Lie group $\mathscr{G}$ in general is a group of operators that depend on a set of continuous parameters such as θ, ϕ, ψ used above to describe rotations. In the notation of operators X acting on vectors $\mathbf{v}$ the binary operation of the group is the successive application of the operators and these must satisfy the following:

If X and $Y \in$ the group $\mathscr{G}$, then

$$XY\mathbf{v} = W\mathbf{v}, \tag{15}$$

where $W \in \mathscr{G}$. The operation is associative in that

$$X(YZ)\mathbf{v} = (XY)Z\mathbf{v}. \tag{16}$$

The group contains the identity operator E which has the property that for all $X \in \mathscr{G}$

$$XE\mathbf{v} = X\mathbf{v}. \tag{17}$$

$\mathscr{G}$ also contains the inverse X^{-1} of every operator X in $\mathscr{G}$, the inverse having the property that

$$XX^{-1}\mathbf{v} = E\mathbf{v}. \tag{18}$$

The simplest non-trivial example of a Lie group is the group of rotation operators in three dimensions considered above.

Lie algebra

The fundamental idea of Sophus Lie's theory of continuous groups is to consider not the whole group but that part of it which lies near the identity, i.e. to consider only infinitesimal transformations. It can be shown that [7] for an infinitesimal transformation each operator of the group can be written in the form

$$X = E + T\,\mathrm{d}\theta, \tag{19}$$

where T is an operator independent of θ.† The set of these operators is known as the Lie algebra and from them all the properties of the group can be discovered. They are shown to obey relations of the form

$$[T_a, T_b] \equiv T_a T_b - T_b T_a = f^c_{ab} T_c, \tag{20}$$

where f^c_{ab} are a set of numerical constants called structure constants. The notation $[T_a, T_b]$ defined in equation (20) is called the 'commutator' of T_a and T_b.

Any set of operators T_i which obey relations like equation (20) form a Lie algebra and are related to the operators of some Lie group by equations like equation (19). They are said to 'generate' the group. If the Lie group contains a subgroup, then there is a subalgebra of the corresponding Lie algebra which generates the subgroup.

Thus, given a group $\mathscr{G}$ depending on n parameters, we get a set of n operators T_i. We can define a vector space spanned by these T_i if we take the scalar product to be zero between different elements T_i and T_j, $i \neq j$. The 'vectors' of the space consist of all formal sums

$$X = \alpha_1 T_1 + \alpha_2 T_2 + \ldots + \alpha_n T_n \tag{21}$$

for which multiplication by a complex number, addition of vectors and taking a scalar product are defined by the usual laws for n-component vectors. For example, if

$$Y = \beta_1 T_1 + \beta_2 T_2 + \ldots + \beta_n T_n$$

then

$$(X, Y) = \alpha_1\beta_1 + \alpha_2\beta_2 + \ldots + \alpha_n\beta_n.$$

Not only do the elements of the space $\mathscr{A}$ spanned by $T_1, T_2, \ldots, T_n$ have all the properties of vectors in a vector space, but there is in addition another binary operation which can be defined in a natural way. This is the operation of taking a commutator. It can be extended to any two elements of $\mathscr{A}$ by the rule

$$[\lambda_1 T_1 + \lambda_2 T_2, T_3] = \lambda_1[T_1, T_3] + \lambda_2[T_2, T_3].$$

† For a mathematically exact account of the relation between a Lie group and a Lie algebra the reader should refer to L. Pontrjagin [8]. For our purposes the above descriptive account is sufficient.

The binary operation of commutation cannot be taken as a group law for $\mathscr{A}$ since, for example

$$[T_1,[T_2,T_3]] \neq [[T_1,T_2],T_3].$$

In general a Lie algebra is defined as a vector space in which there is defined an additional binary operation of multiplication called commutation, which is such that the commutator $[X, Y]$ of two vectors X and Y is a third vector also defined in the space. The commutation operation must also satisfy the following relations:

$$[\lambda_1 X_1+\lambda_2 X_2, Y] = \lambda_1[X_1, Y]+\lambda_2[X_2, Y] \tag{22}$$

$$[X, Y] = -[Y, X] \tag{23}$$

$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]] = 0, \tag{24}$$

where λ_1 and λ_2 are complex numbers.

A comparison of these equations with equation (20) shows that the definition of a commutator as

$$[X, Y] = XY-YX$$

satisfies the defining equations so we have a Lie algebra. However, the definition does not restrict us to commutators of this form. Since λ_1 and λ_2 are complex numbers, the space defined is a ' complex space ' in that any vector in the space may be multiplied by a complex number. A space consisting of the same vectors, but with the restriction that they only be multiplied by real numbers, would be an ordinary Euclidean space where scalar products could be taken in the usual way. The relation (24) is called the Jacobi identity.

It is now possible to give a simple example of a Lie algebra. Consider the set of the vectors of a three-dimensional space and the binary operation of taking a vector product. (As it is a Lie algebra we may call the operation one of commutation.) If X, Y, etc., are vectors in a three-dimensional space and $[X, Y]$ represents a vector product, it is simple to show that the above equations (22)–(24) are true. It follows that the three-dimensional space with the binary operation of taking vector products is a Lie algebra having three generators—a set of vectors which span the space.

Isomorphism

A Lie algebra $\mathscr{A}$ is said to be isomorphic to a Lie algebra $\mathscr{A}'$ if there is a one-to-one correspondence between the basic vectors that

preserves the commutation relations. This means that to each member X of $\mathscr{A}$ there is a member $F(X)$ of $\mathscr{A}'$ such that

$$F([X, Y]) = [F(X), F(Y)]. \tag{25}$$

Since the definition of a Lie algebra is concerned only with the dimension of the space and the binary operation of commutation, it follows that two isomorphic algebras $\mathscr{A}$ and $\mathscr{A}'$ can be treated as one and the same.

Representation

The generators of a Lie algebra, defined as spanning a vector space as above, are isomorphic to various sets of matrices. Each such set is called a ' representation ' of the Lie algebra. In this case, each generator of the algebra has a corresponding square matrix. Each element of every matrix is an explicit number. The size of the matrix, r, is known as its ' dimension '. As all the matrices for any representation must be of the same dimension, r is also the dimension of the representation. For any particular Lie algebra there are many values of r for which representations of the algebra may be found.

The relations between the matrices must be such that the commutation relations are preserved. Thus if the matrix $M(X)$ corresponds to the operator X, then from

$$[X_a, X_b] = f_{ab}^c X_c \tag{26}$$

it follows that

$$M(X_a)M(X_b)-M(X_b)M(X_a) = f_{ab}^c M(X_c), \tag{27}$$

where $M(X_a)M(X_b)$ implies the matrix multiplication of the two matrices and every element of the matrix on the left must be equal to the corresponding element of the matrix on the right. The structure constant f_{ab}^c is of course a number. In general the matrix representing an operator will be designated by the same symbol as the operator.

Consider the example of the Lie algebra which consists of a three-dimensional space and the operation of taking vector products. Let the three operators of the algebra be labelled X, Y and Z. The operator X corresponds to the unit vector along the x-axis, **x**, the operator Y to **y** and the operator Z to **z**. We have as ' commutation relations '

$$\left.\begin{aligned} \mathbf{x} \wedge \mathbf{y} &= \mathbf{z}, \\ \mathbf{y} \wedge \mathbf{z} &= \mathbf{x}, \\ \mathbf{z} \wedge \mathbf{x} &= \mathbf{y}, \end{aligned}\right\} \tag{28}$$

or

$$\left.\begin{aligned} [X, Y] &= Z, \\ [Y, Z] &= X, \\ [Z, X] &= Y. \end{aligned}\right\} \tag{29}$$

There are two very common representations of this algebra, one in two dimensions and one in three dimensions. Here we shall merely state the matrices of these representations so that the concept of a representation is made clear in the reader's mind. In Chapter 4 the method of derivation is described.

First let

$$X_1 = \tfrac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

$$Y_1 = \tfrac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{30}$$

$$Z_1 = \tfrac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

and it is easy to check that when $[X, Y]$ is interpreted as

$$XY - YX \tag{31}$$

the relations (29) hold for these matrices.

Secondly, the same is true for the representation

$$X_2 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$Y_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \tag{32}$$

$$Z_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Thus we see that this particular Lie algebra has a representation in two dimensions and another representation in three dimensions.

It follows that it also has a representation in five dimensions, this being the combination of the two above, i.e.

$$X = \begin{pmatrix} 0 & \frac{i}{2} & 0 & 0 & 0 \\ \frac{i}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Y = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \tag{33}$$

$$Z = \begin{pmatrix} \frac{i}{2} & 0 & 0 & 0 & 0 \\ 0 & -\frac{i}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

Because each of these can be written in the form

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix},$$

$$Y = \begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix}, \tag{34}$$

$$Z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix},$$

where the two-dimensional X_1, Y_1, Z_1 and the three-dimensional X_2, Y_2, Z_2 each satisfy the criteria for being a representation, the five-dimensional representation is said to be ' reducible '. Representations for which this is not so are said to be ' irreducible '.

Simple Lie algebra

If a subset of the algebra has the property that the commutator of any member of the subset with any member of the algebra produces a member of the subset, this subset is called an *ideal.*† If the algebra contains members which are not in the ideal, then it is called a *proper ideal.*

Let the members of a subset be labelled X_{i_1}, X_{i_2}, etc., and the other members of the algebra be labelled Y_{j_1}, Y_{j_2}, etc. The subset is an ideal if

$$[X_i, Y_j] = f_{ij}^k X_k \tag{35}$$

for all i and j. If a Lie algebra has no proper ideals, then it is called a ' simple Lie algebra '.

Consider the case of the unitary algebra generated by the eight operators $T_1, T_2, \ldots, T_8$ and the commutation relations listed in Chapter 2. It can be seen that this algebra is a simple Lie algebra. On the other hand, the algebra generated by the X, Y and Z of our three-dimensional algebra and the eight operators of the unitary algebra is not a simple Lie algebra since commutation relations of the form

$$[X, T_4] \tag{36}$$

are taken to be zero, and zero is always understood to be a member of every algebra. It follows that this algebra contains two proper ideals and is not simple. It is known as the direct sum of the two algebras. The direct sum of two or more non-trivial simple Lie algebras is a semisimple Lie algebra.

Suppose $\mathscr{R}_1, \mathscr{R}_2, \ldots, \mathscr{R}_s$ are s simple Lie algebras. We define the direct sum of them, $\mathscr{R}$, as the vector space spanned by all the elements of $\mathscr{R}_1, \mathscr{R}_2, \ldots, \mathscr{R}_s$ with the binary operation of commutation, where

$$[X, Y] = 0$$

if X and Y belong to different algebras $\mathscr{R}_i$, and

$$[X, Y] = \text{same as in } \mathscr{R}_i$$

if $X \in \mathscr{R}_i$ and $Y \in \mathscr{R}_i$. The direct sum is written

$$\mathscr{R} = \mathscr{R}_1 + \mathscr{R}_2 + \ldots + \mathscr{R}_s$$
$$= \sum_{i=1}^{s} \cdot \mathscr{R}_i.$$

† Sometimes known as an invariant subalgebra.

The direct sum of two Lie algebras is the algebra of the direct product of the corresponding groups.

1.3 Summary of Chapter 1

1. A set is any kind of a collection of entities of any sort. A subset is a collection of some of the entities of the set.

A linear space is a set and the operations of multiplication by a complex number and addition of elements.

A vector space is a linear space and the operation of taking a scalar product.

An algebra is a linear space and a binary operation of multiplication which combines any two members of the set forming a third member of the set.

A group is an algebra which satisfies the axioms:

(i) The binary operation is associative.

(ii) The set contains an identity element.

(iii) For each element in the set there is an inverse element in the set.

A subgroup is a subset of a group which by itself forms a group.

2. All groups of continuous operators are called Lie groups. A Lie algebra is the vector space spanned by the operators generating the infinitesimal transformations of a Lie group, together with the binary operation of commutation.

An isomorphism is a one-to-one mapping of one algebra on another so that the binary operation is maintained.

The generators of a Lie algebra are isomorphic to various sets of matrices known as representations of the algebra. The representations may be reducible or irreducible.

3. An ideal is a subset of a Lie algebra such that the commutation of a member of the subset with any member of the algebra results in a member of the subset.

A simple Lie algebra is a Lie algebra with no proper ideals.

The ' direct sum ' of a number of simple Lie algebras is the algebra consisting of all the members of all the simple Lie algebras.

A direct sum of non-trivial simple Lie algebras is a ' semi-simple' Lie algebra.

CHAPTER 2

General Properties of Simple Lie Algebras

In Chapter 1 a simple Lie algebra was defined as a set and the operations of multiplication by complex numbers, addition of elements, taking a scalar product of elements and taking a commutator of elements. Here we shall view a simple Lie algebra as a space, subalgebras as subspaces and the elements of the algebra as the vectors of the space. The vectors spanning the space can be chosen to correspond to the generators of the simple Lie algebra. From now on the reader will frequently find himself trying to imagine systems of vectors spanning spaces of various dimensions. The structure of a space which is a simple Lie algebra and the properties of the vectors which span the space and generate the simple Lie algebra can be proved [9] to follow directly from the definition of a simple Lie algebra.

2.1 Structure of a simple Lie algebra

The space to which a simple Lie algebra corresponds is defined in Theorem 1.

Theorem 1 [9]

A simple Lie algebra $\mathscr{R}$ is a space $\mathscr{R}$ which can be split into subspaces with each subspace corresponding to a subalgebra in the following way:

$$\mathscr{R} = \mathscr{H} + \sum_{\alpha \in \Sigma}{}^{\cdot} \mathscr{R}^{\alpha}. \tag{1}$$

The $\mathscr{R}^{\alpha}$ are one-dimensional subspaces. It follows that the subalgebras to which they correspond can only contain one linearly independent element apart from the zero element, and therefore satisfy the definition of a subalgebra given in 1.1. The subspace $\mathscr{H}$ is of greater interest and is generally of more than one dimension. There are a set of vectors α, β, etc., characteristic of the algebra $\mathscr{R}$ and known as its ' roots ' in the subspace $\mathscr{H}$. The set of roots is

known as Σ. One of the one-dimensional subalgebras $\mathscr{R}^{\alpha}$ ($\alpha \in \Sigma$ as in equation (1)) corresponds to each root. The roots of the algebra are always such that if α is a root ($\alpha \in \Sigma$) then $-\alpha$ is also a root ($-\alpha \in \Sigma$) but $k\alpha$ is not a root where k is any real number except ± 1. ($k\alpha \bar{\in} \Sigma$ for $k \neq \pm 1$.)

Suppose the subspace $\mathscr{H}$ has m dimensions. It can then be spanned by any set of m linearly independent vectors. For any simple Lie algebra $\mathscr{R}$ there is a subset of the set of roots Σ which is labelled Π ($\Pi \subset \Sigma$) and whose members are known as ' simple roots '. This subset spans the space $\mathscr{H}$ and therefore consists of m linearly independent vectors. These vectors are not necessarily orthogonal. Although they may be taken to be the generators of the algebra $\mathscr{R}$ which correspond to the subalgebra $\mathscr{H}$, it is often more convenient to take linear combinations of these simple roots so that an orthogonal set spanning $\mathscr{H}$ is formed, and call these the generators of $\mathscr{R}$.

The space $\mathscr{R}$ is a complex space—that is any vector in the space may have a complex amplitude. In dealing with roots and simple roots we are only concerned with vectors with real amplitudes, and it is useful to define a space $\mathscr{D}$ called the ' idempotent ' of the algebra $\mathscr{R}$. It will have the same number of dimensions as $\mathscr{H}$ but is a real space in that it is the collection of all vectors of the form

$$\sum_{\alpha \in \Pi} a_{\alpha}\alpha, \tag{2}$$

where the a_{α} are real numbers. This is a Euclidean space in which scalar products may be formed in the usual way. The dimension of $\mathscr{D}$, which is equal to the dimension of $\mathscr{H}$ and to the number of simple roots of the algebra, is called the ' rank ' of the algebra.†

2.2 Classification by simple roots

The simple roots of a simple Lie algebra are fundamental. From the relative lengths of simple roots and the angles between them, the lengths and directions of the remaining roots can be calculated by a simple procedure described in 2.3. All the properties of the algebra will be seen to depend on the roots. For example, the structure

† It will be seen later that the generators of the algebra which correspond to the simple roots are the only ones represented by diagonal matrices. Thus the rank of the group or algebra is the number of generators which can be simultaneously diagonalised and this, in quantum mechanics, is the number of conserved quantities.

constants calculated in 2.4, the dimensions of representations in 3.2 and the actual matrix elements of the representations in Chapters 4, 5 and 6. It follows that simple Lie algebras can be completely specified by specifying the lengths and directions of their simple roots.

In general a set of linearly independent vectors is not the set of simple roots of a simple Lie algebra. Theorem 2 describes the properties a set of vectors must have if it is to be a set of simple roots of a simple Lie algebra.

Theorem 2 [9]
A set of linearly independent vectors spanning a space is a set of simple roots of a simple Lie algebra if and only if the scalar product of any two of the vectors is either zero, or is equal to minus an integer times half the length of either of the vectors.

That is, if $\alpha \in \Pi$ and $\beta \in \Pi$,

$$(\alpha, \beta) = -N\frac{(\alpha,\alpha)}{2} = -M\frac{(\beta,\beta)}{2}, \tag{3}$$

where N and M are positive integers or zero.

This condition restricts both the angles between the simple roots and their relative lengths. Put

$$(\alpha,\alpha) = \lambda, \qquad (\beta,\beta) = c\lambda, \tag{4}$$

where λ and c are real numbers. Then

$$(\alpha,\beta) = \sqrt{c}\lambda\cos\theta = -\frac{N\lambda}{2} = -\frac{cM\lambda}{2}, \tag{5}$$

where θ is the angle between the simple roots α and β in the Euclidean idempotent $\mathscr{D}$. Since

$$-1 \leqslant \cos\theta \leqslant +1 \tag{6}$$

and

$$\left.\begin{array}{ll} \cos\theta = 1 & \text{only when } \alpha = \beta, \\ \cos\theta = -1 & \text{only when } \alpha = -\beta, \end{array}\right\} \tag{7}$$

(since $k\alpha \notin \Sigma$ for $k \neq \pm 1$), equation (5) states

$$-1 \leqslant -\frac{N}{2\sqrt{c}} \leqslant 1 \tag{8}$$

and

$$-1 \leqslant -\frac{M\sqrt{c}}{2} \leqslant 1. \tag{9}$$

Rewriting equations (8) and (9) as

$$\left|\frac{N}{\sqrt{c}}\right| \leqslant 2 \tag{10}$$

$$|M\sqrt{c}| \leqslant 2 \tag{11}$$

gives

$$|MN| \leqslant 4, \tag{12}$$

where N and M are positive integers. Excluding the case where $\alpha = \pm\beta$ excludes the equal sign in (12) so that

$$|NM| < 4. \tag{13}$$

It follows that when

$$\left.\begin{array}{ll} & M = 1, \quad N = 1, 2 \text{ or } 3 \\ \text{when} & \\ & M = 2, \quad N = 1, \\ \text{and when} & \\ & M = 3, \quad N = 1. \end{array}\right\} \tag{14}$$

From equation (5)

$$c = \frac{N}{M} \tag{15}$$

and

$$\cos\theta = -\tfrac{1}{2}\sqrt{NM}$$

$$= -\tfrac{1}{2} \text{ or } -\frac{\sqrt{2}}{2} \text{ or } -\frac{\sqrt{3}}{2} \tag{16}$$

giving

$$\theta = 120° \text{ or } 135° \text{ or } 150°. \tag{17}$$

If the scalar product is zero

$$N = M = 0.$$

Then

$$\cos\theta = 0 \tag{18}$$

giving

$$\theta = 90°. \tag{19}$$

Thus it follows from Theorem 2 that a set of vectors can only be the simple roots of a simple Lie algebra, if the angles between them are either 90°, 120°, 135° or 150°.

Another result of Theorem 2 is that for any particular simple Lie algebra the simple roots either all have the same length or all have one of two lengths.

As a result of this restriction on the lengths of and angles between the simple roots, a shorthand method of specifying a set of simple roots and thereby specifying a simple Lie algebra has been developed [9]. Each simple root is represented by a small circle. If the simple root is a long one, the circle is left white—○; if it is a short one, the circle is made black—●. The angle between any two roots is denoted by the number of lines joining the circles which represent them. If they are not joined they are orthogonal. If the angle between them is 120°, 135° or 150° they will be joined by one, two or three lines respectively. The resulting diagrams are known as Schouten diagrams. As a result of the restrictions imposed by Theorem 2 no simple root forms an angle other than 90° with more than three other simple roots, so the system leads to no confusion. In fact it can be shown that, as a result of Theorem 2, there exist only the four infinite series of simple Lie algebras shown in Fig. 2.1 and the five ' exceptional ' algebras shown in Fig. 2.2.

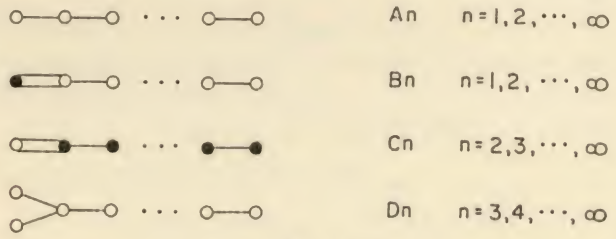


Fig. 2.1

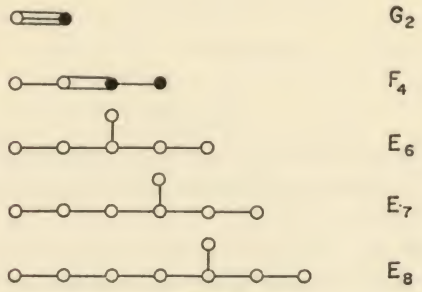


Fig. 2.2

In Fig. 2.1 the algebras A_n ($n = 1, 2, \ldots, \infty$) are known as the unitary unimodular algebras and consist of all those transformations of a $(n+1)$-dimensional space whose traces are zero. They correspond to the group known as SU(r) where $r = n+1$. The B_n ($n = 1, 2, \ldots, \infty$) are the orthogonal algebras of odd dimension. They describe rotations in spaces of $2n+1$ dimensions and correspond to the group called O(l), where $l = 2n+1$. The D_n ($n = 3, 4, \ldots, \infty$) are the orthogonal algebras of even dimensions—O(l) where $l = 2n$. They describe rotations in spaces of $2n$ dimensions. The algebra denoted by D_2 corresponding to O(4) describes rotations in four dimensions and is excluded from the list as it is not a simple algebra, but is semisimple being the direct sum of B_1 with itself. The C_n ($n = 2, 3, \ldots, \infty$) are the symplectic algebras. They are matrices in a complex space of $2n$ dimensions and correspond to the group known as U(l), where $l = 2n$. The suffix n gives the rank of the algebra, and therefore corresponds to the number of simple roots.

Algebras which have identical systems of simple roots are identical. It follows that

$$\left.\begin{aligned} A_1 &\equiv B_1, \\ A_3 &\equiv D_3, \\ B_2 &\equiv C_2. \end{aligned}\right\} \tag{20}$$

2.3 Derivation of roots from simple roots

We have seen that the set of roots of a simple Lie algebra is a set of vectors in a space $\mathscr{H}$. The set of roots, Σ, contains a set of simple roots, Π, which form a linearly independent set of vectors spanning the space $\mathscr{H}$. It follows that the roots can be expressed in terms of the simple roots, i.e.

$$\mu = \sum_{\alpha \in \Pi} a_\alpha^\mu \alpha, \qquad \mu \in \Sigma. \tag{21}$$

For any particular root μ the a_α^μ are either all positive or all negative. The ' power ' of a root μ, $\delta(\mu)$, is defined by equation (22)

$$\delta(\mu) = \sum_{\alpha \in \Pi} a_\alpha^\mu. \tag{22}$$

If it is positive, the root is known as a positive root, if negative, as

a negative root. The set of all the positive roots is labelled Σ_+ so that

$$\left.\begin{array}{r}\mu \in \Sigma_+ \\ \delta(\mu) > 0.\end{array}\right\} \quad \text{if and only if} \tag{23}$$

The simple roots are the roots with unit power, i.e.

$$\delta(\mu) = 1 \qquad \text{for } \mu \in \Pi. \tag{24}$$

Since if $\mu \in \Sigma$ then $-\mu \in \Sigma$ it is only necessary to find the set of positive roots in order to discover the set of all the roots.

The following theorem gives the rule for calculating the roots of a simple Lie algebra from its simple roots [9].

Theorem 3

If μ is a positive root of a simple Lie algebra, then

$$\mu + \alpha \in \Sigma_+ \tag{25}$$

where α is a simple root ($\alpha \in \Pi$) if and only if

$$\frac{2(\mu, \alpha)}{(\alpha, \alpha)} - P(\mu, \alpha) < 0, \tag{26}$$

where $P(\mu, \alpha)$ is the integer defined by

$$\left.\begin{array}{r}\mu - P(\mu,\alpha)\alpha \in \Sigma_+ \\ \mu - [P(\mu,\alpha)+1]\alpha \notin \Sigma_+.\end{array}\right\} \tag{27}$$

It follows that if the positive roots are calculated systematically in order of increasing power, the value of P will always be known. Some examples will clarify the use of Theorem 3.

Example 1

First consider the simplest algebra, that with only one simple root, i.e. the algebra known as B_1 or A_1. This is the algebra mentioned earlier which describes rotations in three dimensions. Its Schouten diagram consists of one circle:

●

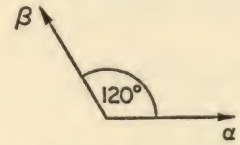
representing the one simple root α. The idempotent is one-dimensional. Since there is only one simple root, there are only two roots, α and $-\alpha$ (since $k\alpha \notin \Sigma$ for $k \neq \pm 1$). The algebra is therefore spanned by three vectors. One in the space $\mathscr{H}$ corresponding to α and labelled F_α, two in the one-dimensional subspaces corresponding to the two roots α and $-\alpha$ and labelled E_α and $E_{-\alpha}$.

Example 2

All the other algebras entail the use of Theorem 3 to find their roots. Consider the algebra A_2 corresponding to the group SU(3) which is of significance in elementary particle physics. Its Schouten diagram is

○—○

representing two simple roots of equal length with an angle of 120° between them. The idempotent is two-dimensional so the simple root system can be represented on paper, see Fig. 2.3.



Simple roots of algebra A_2 or SU(3)

Fig. 2.3

Labelling the vectors α and β we have

$$\left.\begin{array}{r}(\alpha, \alpha) = (\beta, \beta) = \lambda, \\ (\alpha, \beta) = -\tfrac{1}{2}\lambda,\end{array}\right\} \quad \text{and} \tag{28}$$

where λ is a real constant defining the length of the vectors. Applying Theorem 3:

$$\alpha + \beta \in \Sigma_+$$

if and only if

$$\frac{2(\alpha, \beta)}{(\beta, \beta)} - P(\alpha, \beta) < 0. \tag{29}$$

Since $\alpha \in \Sigma_+$ and $\alpha - \beta \notin \Sigma_+$

$$P(\alpha, \beta) = 0 \tag{30}$$

and therefore since

$$\frac{2(\alpha, \beta)}{(\beta, \beta)} = -1 < 0 \tag{31}$$

it follows that

$$\alpha + \beta \in \Sigma_+. \tag{32}$$

Since $\alpha + \beta \in \Sigma_+$ we must now consider the possibilities of $\alpha + 2\beta$ and $2\alpha + \beta$ being positive roots. Using Theorem 3 again:

$$\alpha + 2\beta \in \Sigma_+$$

if and only if

$$\frac{2(\alpha+\beta, \beta)}{(\beta, \beta)} - P(\alpha+\beta, \beta) < 0. \tag{33}$$

Since

$$\left.\begin{aligned} &\alpha + \beta - \beta \in \Sigma_+ \\ &\alpha + \beta - 2\beta \notin \Sigma_+ \end{aligned}\right\} \tag{34}$$

and

it follows that

$$P(\alpha+\beta, \beta) = 1 \tag{35}$$

and, since

$$\frac{2(\alpha+\beta, \beta)}{(\beta, \beta)} - 1 = 0 \not< 0, \tag{36}$$

$$\alpha + 2\beta \notin \Sigma_+. \tag{37}$$

By a similar procedure we find

$$2\alpha + \beta \notin \Sigma_+.$$

It follows that

$$\Sigma_+ \equiv \{\alpha, \beta, \alpha+\beta\} \tag{38}$$

and

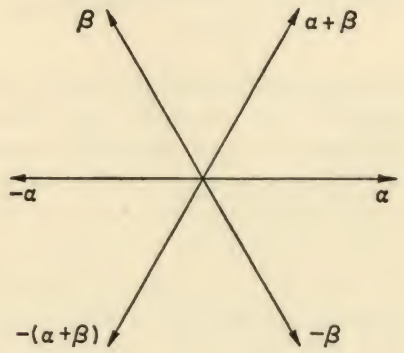
$$\Sigma \equiv \{\alpha, \beta, \alpha+\beta, -\alpha, -\beta, -(\alpha+\beta)\}. \tag{39}$$

The algebra A_2, the algebra of the group SU(3), has eight generators. Two correspond to the two linearly independent vectors in the two-dimensional space $\mathscr{H}$. (These may be taken to correspond to

the simple roots α and β, i.e. F_α and F_β or they may be orthogonal, for instance corresponding to α and $\alpha+2\beta$ since $(\alpha, \alpha+2\beta) = 0$). One corresponds to each of the six one-dimensional subspaces $\mathscr{R}^\alpha$ where $\alpha \in \Sigma$. These are labelled

$$E_\alpha, \quad E_\beta, \quad E_{\alpha+\beta}, \quad E_{-\alpha}, \quad E_{-\beta}, \quad E_{-(\alpha+\beta)}.$$

The two-dimensional root diagram for the algebra A_2 is shown in Fig. 2.4.



System of roots of algebra A_2 or SU(3)

Fig. 2.4

Example 3

The root diagram for the exceptional algebra G_2 is more complicated than that of the algebra A_2 although still in two-dimensions. The Schouten diagram for G_2 is

from which it follows that there are two simple roots, a long one, α, and a short one, β, with an angle of 150° between them.

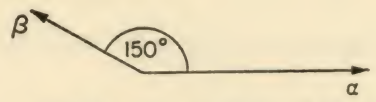
Since

$$\left.\begin{aligned} &\cos\theta = -\frac{\sqrt{3}}{2}, \\ &(\beta, \beta) = \lambda, \\ &(\alpha, \alpha) = n\lambda, \end{aligned}\right\} \tag{40}$$

where λ defines the length of β and n is an integer, it follows that

$$(\alpha, \beta) = -\tfrac{1}{2}\sqrt{3n}\lambda = -\tfrac{3}{2}\lambda \tag{41}$$

in order that Theorem 2 is satisfied, so $n = 3$. The simple root diagram is therefore as in Fig. 2.5.



Simple root system of G_2

Fig. 2.5

The application of Theorem 3 to this and to all the more complicated algebras becomes very involved. The following tabular method of finding the roots is helpful. In the first column the vector under consideration is written. In the second the value of the scalar product term of relation (26) is written and in the third the value of the P term. In the last column it is noted whether or not the vector in the first column is a root. If it is, all the vectors resulting from the addition of one simple root to this vector are listed at the bottom of the first column. Note that vectors like $2\alpha+2\beta$ are not considered when $\alpha+\beta$ is already established as a root, since for $\mu \in \Sigma$, $k_\mu \bar{\in} \Sigma$ for $k \neq \pm 1$.

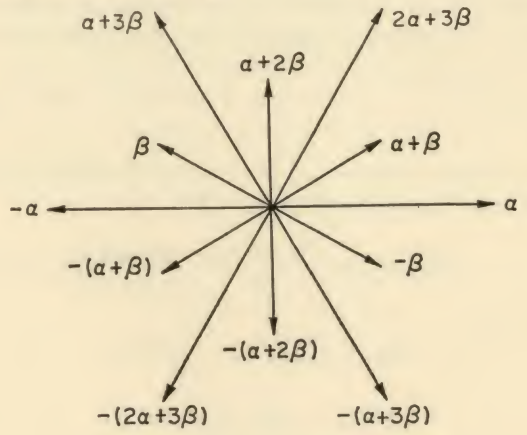
This method is used below to find the roots of G_2.

Vector	*Scalar product*	*P-term*	*Conclusion*
$\alpha+\beta$	-3	0	$\in \Sigma_+$
$\alpha+2\beta$	-1	1	$\in \Sigma_+$
$2\alpha+\beta$	1	1	No
$\alpha+3\beta$	1	2	$\in \Sigma_+$
$\alpha+4\beta$	3	3	No
$2\alpha+3\beta$	-1	0	$\in \Sigma_+$

Adding either simple root to $2\alpha+3\beta$ results in an integer times a root already established so neither need be considered and the system of positive roots is

$$\Sigma_+ \equiv \{\alpha, \beta, \alpha+\beta, \alpha+2\beta, \alpha+3\beta, 2\alpha+3\beta\}. \tag{42}$$

The root diagram is as in Fig. 2.6.



Root system of G_2

Fig. 2.6

The algebra G_2 has 14 generators. Two in the space $\mathscr{H}$ and one corresponding to each of the twelve one-dimensional subspaces $\mathscr{R}^\alpha$ ($\alpha \in \Sigma$).

2.4 Commutation relations and structure constants

Once the roots of a simple Lie algebra have been found, the commutation relations can be calculated. The form of the commutation relations naturally depends upon the linear combinations of operators from which they are made up. The generators fall into two groups: those spanning the subspace $\mathscr{H}$ and labelled F_μ (where μ is the vector in $\mathscr{H}$ to which the operator F_μ corresponds) and those in the one-dimensional subspaces $\mathscr{R}^\alpha$ ($\alpha \in \Sigma$) which are labelled E_α. In Theorem 4 we give the commutation relations of the E_αs and the F_μs. We shall then find a linearly independent set of combinations of these generators which has the property that the structure constants f^t_{rs} defined by

$$[T_r, T_s] = f^t_{rs} T_t \tag{43}$$

are totally antisymmetric. That is to say, the f^t_{rs} changes sign when any two of its indices are interchanged. It will be clear from examination of Theorem 4 that the structure constants will only have this

property of total antisymmetry when the correct linear combinations of the generators are taken. In future, references to the structure constants of a simple Lie algebra mean this set of totally antisymmetric constants.

Theorem 4

The commutation relations of the operators generating a simple Lie algebra may be listed as follows [**10**]:

$$[E_\alpha, E_\beta] = \begin{cases} N_{\alpha,\beta} E_{\alpha+\beta}, & \alpha+\beta \in \Sigma \\ 0, & \alpha+\beta \notin \Sigma \end{cases} \tag{44}$$

for α and $\beta \in \Sigma$.

$$[E_\rho, E_{-\rho}] = F_\rho = \sum_{\alpha \in \Pi} a_\alpha^\rho F_\alpha, \tag{45}$$

$$[F_\mu, F_\nu] = 0, \qquad \mu, \nu \in \Pi, \tag{46}$$

$$[F_\mu, E_\nu] = -(\mu, \nu) E_\nu. \tag{47}$$

The $N_{\alpha,\beta}$ in equation (44) is given by

$$N_{\alpha,\beta}^2 = \frac{(P(\alpha,\beta)+1)}{2} Q(\alpha,\beta)(\beta,\beta) \tag{48}$$

where

$$\frac{2(\alpha,\beta)}{(\beta,\beta)} = P(\alpha,\beta) - Q(\alpha,\beta) \tag{49}$$

and $P(\alpha, \beta)$ and $Q(\alpha, \beta)$ are such that

$$\left.\begin{aligned} &\alpha - P(\alpha,\beta)\beta \in \Sigma \quad \text{while } \alpha - [P(\alpha,\beta)+1]\beta \notin \Sigma \\ \text{and} \\ &\alpha + Q(\alpha,\beta)\beta \in \Sigma \quad \text{while } \alpha + [Q(\alpha,\beta)+1]\beta \notin \Sigma. \end{aligned}\right\} \tag{50}$$

The choice of vectors spanning $\mathscr{R}$ is such that

$$N_{\alpha,\beta}^2 = N_{\beta,\alpha}^2 = N_{-\beta,-\alpha}^2, \text{ etc.}, \tag{51}$$

and the signs should be chosen such that [**10**]

$$N_{\alpha,\beta} = -N_{\beta,\alpha} = N_{-\alpha,-\beta} = -N_{-\beta,-\alpha}. \tag{52}$$

The scalar product in equations (47), (48) and (49) is the scalar product of the vectors in the idempotent $\mathscr{D}$.

It can be seen from the above relations that all the vectors (or operators) in $\mathscr{R}$ can be multiplied by a constant factor, real or

imaginary, without affecting the structure of the algebra $\mathscr{R}$. (The structure constants will also be multiplied by the same factor.) However, once the length of one of the vectors (or operators) is fixed, the lengths of all the others are determined.

It is clear from Theorem 4 that a simple Lie algebra can be described by a set of structure constants which are not totally antisymmetric. Putting

$$T_a = F_\mu \quad \text{and} \quad T_b = E_\nu \tag{53}$$

in (47), the equation takes the form

$$\begin{aligned} [T_a, T_b] &= (-\mu, \nu) T_b \\ &= f_{ab}^b T_b \end{aligned} \tag{54}$$

with

$$f_{ab}^b \neq 0. \tag{55}$$

This cannot be so for a totally antisymmetric constant. It follows that the set consisting of the operators corresponding to the simple roots, and to the vectors in the subspaces $\mathscr{R}^\alpha$ ($\alpha \in \Sigma$) is not the desired set.

First we find a set, Γ say, of orthogonal vectors which span the idempotent. The roots can then be expressed in terms of these vectors, i.e.

$$\alpha = \sum_{\sigma \in \Gamma} \frac{(\alpha,\sigma)}{(\sigma,\sigma)} \sigma, \quad \alpha \in \Sigma, \tag{56}$$

where

$$(\sigma, \rho) = 0, \quad \sigma, \rho \in \Gamma.$$

The operators directly corresponding to these vectors are labelled F_ρ ($\rho \in \Gamma$). Putting

$$\left.\begin{aligned} U_\alpha &= E_\alpha + E_{-\alpha}, & \alpha \in \Sigma, \\ V_\alpha &= i(E_\alpha - E_{-\alpha}), & \alpha \in \Sigma, \\ H_\rho &= iF_\rho, & \rho \in \Gamma, \end{aligned}\right\} \tag{57}$$

and substituting in equations (44)–(47) we find

$$[U_\alpha, U_\beta] = N_{\alpha,\beta} U_{\alpha+\beta} + N_{\alpha,-\beta} U_{\alpha-\beta}, \tag{58}$$

$$[U_\alpha, V_\beta] = N_{\alpha,\beta} V_{\alpha+\beta} - N_{\alpha,-\beta} V_{\alpha-\beta}, \tag{59}$$

$$[V_\alpha, V_\beta] = -N_{\alpha,\beta} U_{\alpha+\beta} + N_{\alpha,-\beta} U_{\alpha-\beta}, \tag{60}$$

$$[U_\alpha, V_\alpha] = -2 \sum_{\rho \in \Gamma} a_\alpha^\rho H_\rho, \tag{61}$$

$$[H_\rho, U_\alpha] = -(\rho, \alpha) V_\alpha, \tag{62}$$

$$[H_\rho, V_\alpha] = (\rho, \alpha) U_\alpha, \tag{63}$$

$$[H_\sigma, H_\rho] = 0. \tag{64}$$

The a_α^ρ in equation (61) are the same as those in

$$\alpha = \sum_{\rho \in \Gamma} a_\alpha^\rho \rho, \quad \alpha \in \Sigma, \tag{65}$$

and taking the scalar product of α with σ where $\sigma \in \Gamma$, it follows from the orthogonality of the vectors in Γ that

$$a_\alpha^\rho = \frac{(\alpha, \rho)}{(\rho, \rho)}. \tag{66}$$

Therefore, if the vectors in Γ are normalised so that

$$(\rho, \rho) = 2, \qquad \text{all } \rho \in \Gamma, \tag{67}$$

equation (61) becomes

$$[U_\alpha, V_\alpha] = -\sum_{\rho \in \Gamma} (\alpha, \rho) H_\rho \tag{68}$$

and examination of equations (68), (62) and (63) shows that for the chosen linear combinations of operators, all structure constants involving diagonal matrices are totally antisymmetric.

It remains to prove that

$$N_{\alpha,\beta}^2 = N_{\alpha+\beta,-\alpha}^2 = N_{\alpha+\beta,-\beta}^2, \tag{69}$$

and, on taking the square root, to choose the correct sign.

The definitions of the $N_{\alpha,\beta}^2$ are given in equations (48)–(50). Using these we have

$$N_{\alpha,\beta}^2 = \frac{P(\alpha,\beta)+1}{2} Q(\alpha,\beta)(\beta,\beta), \tag{70}$$

$$N_{\alpha+\beta,-\beta}^2 = \frac{P(\alpha+\beta,-\beta)+1}{2} Q(\alpha+\beta,-\beta)(\beta,\beta). \tag{71}$$

The Ps and Qs are defined by

$$\alpha - P(\alpha,\beta)\beta \in \Sigma \quad \text{while} \quad \alpha - [P(\alpha,\beta)+1]\beta \notin \Sigma \tag{72}$$

$$\alpha + Q(\alpha,\beta)\beta \in \Sigma \quad \text{while} \quad \alpha + [Q(\alpha,\beta)+1]\beta \notin \Sigma \tag{73}$$

also

$$\alpha+\beta+P(\alpha+\beta,-\beta)\beta \in \Sigma \quad \text{while} \quad \alpha+\beta+[P(\alpha+\beta,-\beta)+1]\beta \notin \Sigma \tag{74}$$

$$\alpha+\beta-Q(\alpha+\beta,-\beta)\beta \in \Sigma \quad \text{while} \quad \alpha+\beta-[Q(\alpha+\beta,-\beta)+1]\beta \notin \Sigma. \tag{75}$$

From equations (72) and (75)

$$P(\alpha,\beta) = Q(\alpha+\beta,-\beta)+1 \tag{76}$$

and from equations (73) and (74)

$$Q(\alpha,\beta) = P(\alpha+\beta,-\beta)+1. \tag{77}$$

Substituting these in equations (70) and (71)

$$N_{\alpha,\beta}^2 = \tfrac{1}{2} Q(\alpha+\beta,-\beta) Q(\alpha,\beta)(\beta,\beta) \tag{78}$$

$$\begin{aligned} N_{\alpha+\beta,-\beta}^2 &= \tfrac{1}{2} Q(\alpha,\beta) Q(\alpha+\beta,-\beta)(\beta,\beta) \\ &= N_{\alpha,\beta}^2. \end{aligned} \tag{79}$$

The second equality in equation (69) can be proved in a similar way, and completes the proof of the total antisymmetry of the structure constants of any simple Lie algebra when the operators are taken in the combinations given by U_α, V_α and H_ρ ($\alpha \in \Sigma$, $\rho \in \Gamma$).

Some examples of calculations of totally antisymmetric structure constants will clarify the use of Theorem 4.

Example 4

First we shall take the simplest algebra, B_1. We have seen that this has one simple root, α, and three generators. Take the linear combinations H_α, U_α and V_α. Applying Theorem 4 we have:

$$\left.\begin{aligned} [U_\alpha, V_\alpha] &= -\tfrac{1}{2} H_\alpha, \\ [H_\alpha, U_\alpha] &= -\tfrac{1}{2} V_\alpha, \\ [H_\alpha, V_\alpha] &= \tfrac{1}{2} U_\alpha. \end{aligned}\right\} \tag{80}$$

Putting

$$\left.\begin{aligned} T_1 &= 2U_\alpha, \\ T_2 &= 2H_\alpha, \\ T_3 &= 2V_\alpha, \end{aligned}\right\} \tag{81}$$

these become

$$\left.\begin{aligned} [T_1, T_2] &= T_3, \\ [T_2, T_3] &= T_1, \\ [T_3, T_1] &= T_2, \end{aligned}\right\} \tag{82}$$

and we see that this is an algebra already met with in Chapter 1. The T_1, T_2, T_3 here correspond to the X, Y, Z of Chapter 1. The totally antisymmetric structure constant is unity, i.e.

$$f_1{}^3{}_2 = f_3{}^2{}_1 = f_2{}^1{}_3 = 1. \tag{83}$$

Example 5

For our next example we take the second simplest algebra, namely A_2. This is an algebra which is of significance in elementary particle physics. Its generators are the set

$$A_2 \equiv \{H_\alpha, H_\sigma, U_\alpha, V_\alpha, U_\beta, V_\beta, U_{\alpha+\beta}, V_{\alpha+\beta}\}, \tag{84}$$

where σ is such that

$$(\alpha, \sigma) = 0 \tag{85}$$

and

$$(\alpha, \alpha) = (\sigma, \sigma) = 2. \tag{86}$$

It follows that the λ of equation (28) is equal to two and

$$\sigma = \frac{1}{\sqrt{3}}(\alpha + 2\beta). \tag{87}$$

As a result of the total antisymmetry of the structure constants for the correct linear combinations of operators, we need only consider two types of commutation relations; those of the form

$$[H_\sigma, U_\rho] = -(\sigma, \rho)V_\rho \tag{88}$$

for $\sigma \in \Gamma$ and $\rho \in \Sigma$ and those containing only Us and Vs.

The first sort can be written down immediately using

$$\left.\begin{aligned} (\alpha, \alpha) = (\beta, \beta) &= 2 \\ (\alpha, \beta) &= -1 \end{aligned}\right\} \tag{89}$$

$$\left.\begin{aligned} [H_\alpha, U_\alpha] &= -2V_\alpha, \\ [H_\alpha, U_\beta] &= V_\beta, \\ [H_\alpha, U_{\alpha+\beta}] &= -V_{\alpha+\beta}, \\ [H_\sigma, U_\alpha] &= 0, \\ [H_\sigma, U_\beta] &= -\sqrt{3}V_\beta, \\ [H_\sigma, U_{\alpha+\beta}] &= -\sqrt{3}V_{\alpha+\beta}. \end{aligned}\right\} \tag{90}$$

For the algebra A_2 there is only one relation between the roots leading to commutation relations of the second sort. That is the sum of the roots α and β to form the root $\alpha+\beta$. Thus the remaining commutation relations are:

$$\left.\begin{aligned} [U_\alpha, U_\beta] &= N_{\alpha,\beta}U_{\alpha+\beta}, \\ [U_\alpha, V_\beta] &= N_{\alpha,\beta}V_{\alpha+\beta}, \\ [V_\alpha, V_\beta] &= -N_{\alpha,\beta}U_{\alpha+\beta}, \\ [U_\beta, V_\alpha] &= N_{\beta,\alpha}V_{\alpha+\beta}. \end{aligned}\right\} \tag{91}$$

It should be noted that there are always four separate structure constants resulting from each such relation between the roots. Since

$$N^2_{\alpha,\beta} = \frac{[P(\alpha,\beta)+1]}{2}Q(\alpha,\beta)(\beta,\beta), \tag{92}$$

and

$$\frac{2(\alpha,\beta)}{(\beta,\beta)} = 1, \tag{93}$$

while

$$P(\alpha,\beta) = 0, \tag{94}$$

and

$$Q(\alpha,\beta) = 1, \tag{95}$$

we have

$$N^2_{\alpha,\beta} = 1 \tag{96}$$

and

$$N_{\alpha,\beta} = \pm 1. \tag{97}$$

Choosing the positive sign and putting

$$\left.\begin{aligned} H_\alpha &= 2T_1 \\ H_\sigma &= 2T_2 \\ U_\alpha &= 2T_3 \\ V_\alpha &= 2T_4 \\ U_\beta &= 2T_5 \\ V_\beta &= 2T_6 \\ U_{\alpha+\beta} &= 2T_7 \\ V_{\alpha+\beta} &= 2T_8 \end{aligned}\right\} \tag{98}$$

we find for the structure constants [1]

$$\left.\begin{aligned} f_1{}^4{}_3 &= -1, \\ f_1{}^6{}_5 &= \tfrac{1}{2}, \\ f_1{}^8{}_7 &= -\tfrac{1}{2}, \\ f_2{}^6{}_5 &= -\frac{\sqrt{3}}{2}, \\ f_2{}^8{}_7 &= -\frac{\sqrt{3}}{2}, \\ f_3{}^7{}_5 &= \tfrac{1}{2}, \\ f_3{}^8{}_6 &= \tfrac{1}{2}, \\ f_4{}^7{}_6 &= -\tfrac{1}{2}, \\ f_4{}^8{}_5 &= \tfrac{1}{2}. \end{aligned}\right\} \qquad (99)$$

2.5 Summary of Chapter 2

1. Simple Lie algebras can be classified by means of diagrams. Any simple Lie algebra corresponds to one of the diagrams in Fig. 2.7. The first four diagrams are infinite series of diagrams. Everything about the algebra can be calculated from its diagram.

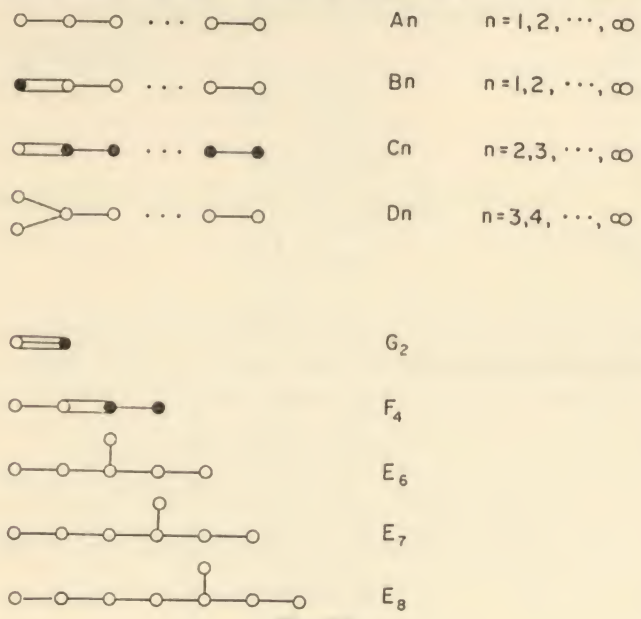


Fig. 2.7

The interpretation of the diagrams is the following: to each spot corresponds a vector. The vector is long if the spot is white, and short if the spot is black. The angle between any two of these vectors is determined by the number of lines joining the two spots. If they are not joined directly they are orthogonal. If they are joined by one, two or three lines, the angle is 120°, 135° or 150° respectively. The lengths of the vectors are determined by the condition that the scalar product of any two must be equal to minus an integer times half the square of one of the vectors. These vectors are known as simple roots and their lengths and orientations determine the properties of the Lie algebra.

2. From the simple roots a set of vectors called roots is derived. This includes the simple roots and all those vectors satisfying the following condition:

If μ is a root then $\mu+\alpha$ is also a root, where α is a simple root if and only if

$$\frac{2(\mu,\alpha)}{(\alpha,\alpha)} - P(\mu,\alpha) < 0$$

where $P(\mu,\alpha)$ is the integer defined by the condition

$$\mu - P(\mu,\alpha)\alpha \text{ is a root}$$

but

$$\mu - [P(\mu,\alpha)+1]\alpha \text{ is not a root.}$$

The negatives of all the vectors found in this way are also roots.

The generators of the Lie algebra can be divided into two categories. There is a generator E_μ corresponding to every root and whose properties depend on the root. In addition there is a generator F_μ corresponding to each simple root α.

3. The commutation relations of the operators generating a simple Lie algebra are as follows:

$$[E_\mu, E_\nu] = \begin{cases} N_{\mu,\nu} E_{\mu+\nu}, & \mu+\nu \in \Sigma, \\ 0, & \mu+\nu \notin \Sigma, \end{cases}$$

$$\mu, \nu \in \Sigma.$$

$$[E_\mu, E_{-\mu}] = \sum_{\alpha\in\Pi} a_\alpha^\mu F_\alpha$$

$$\mu \in \Sigma, \qquad \mu = \sum_{\alpha\in\Pi} a_\alpha^\mu \alpha.$$

$$[F_\alpha, F_\beta] = 0,$$
$$\alpha, \beta \in \Pi.$$
$$[F_\alpha, E_\mu] = -(\alpha, \mu)E_\mu,$$
$$\alpha \in \Pi, \qquad \mu \in \Sigma.$$

In the above (α, μ) is the scalar product of the vectors α and μ, and $N_{\mu,\nu}$ is given by

$$N^2_{\mu,\nu} = \frac{P(\mu,\nu)+1}{2} Q(\mu,\nu)(\nu,\nu)$$

where

$$\frac{2(\mu,\nu)}{(\nu,\nu)} = P(\mu,\nu) - Q(\mu,\nu),$$

while

$$\mu - P(\mu,\nu)\nu \in \Sigma \quad \text{and} \quad \mu - [P(\mu,\nu)+1]\nu \notin \Sigma$$

and

$$\mu + Q(\mu,\nu)\nu \in \Sigma \quad \text{and} \quad \mu + [Q(\mu,\nu)+1]\nu \notin \Sigma$$

with

$$P(\mu,\nu) > 0 \quad \text{and} \quad Q(\mu,\nu) > 0.$$

PROBLEMS 2

1. By calculating the roots of the algebra A_3 show that it has 15 generators.
2. Draw the simple root diagram and the root diagram for the algebra B_2.
3. How many operators generate the algebra B_3?
4. Find the structure constants of the algebra B_2.
5. Find the structure constants of the algebra A_3.

CHAPTER 3

Irreducible Representations of Simple Lie Algebras

The operators of a simple Lie algebra may be represented† by $n \times n$ matrices. These operate on vectors in an n-dimensional space, transforming one vector of the space into another. This n-dimensional space is entirely different from the space $\mathscr{R}$ in which the simple Lie algebra $\mathscr{R}$ is defined. The existence of an n-dimensional representation of $\mathscr{R}$ necessitates the existence of the n-dimensional ' representation space '. It is at this point that the mathematics of simple Lie algebras becomes significant in the theory of elementary particles. The assumption is made that, in the same way as the vectors spanning the space $\mathscr{R}$ are isomorphic to the generators of a simple Lie algebra, so the vectors spanning the representation space $\mathscr{A}_n$ are isomorphic to a set of elementary particles. It is assumed that the interactions of the elementary particles are invariant to a first approximation under the operators forming the simple Lie group.

3.1 Classification of irreducible representations by their greatest weights

The dimensions of the representations of any particular simple Lie algebra are restricted. To every possible irreducible representation of a simple Lie algebra, there corresponds a vector λ in the idempotent $\mathscr{D}$ which is called the ' greatest weight ' of the representation. From this greatest weight all the properties of the representation including the dimension can be calculated. The following theorem restricts the vectors in the idempotent which can be greatest weights of representations and permits a classification of representations by means of Schouten diagrams.

† See Chapter 1.

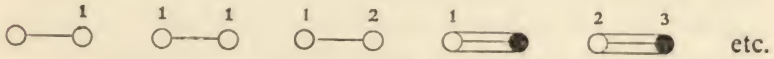
Theorem 5 [11]

In order that an element λ of the idempotent $\mathscr{D}$ be the greatest weight of some irreducible representation it is necessary and sufficient that all the numbers

$$\lambda_\alpha = 2\frac{(\lambda,\alpha)}{(\alpha,\alpha)}, \quad \alpha \in \Pi \tag{1}$$

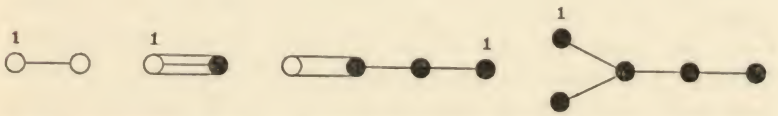
be non-negative integers.

The irreducible representation corresponding to some greatest weight λ can then be completely specified by drawing the Schouten diagram of the simple Lie algebra, and over each circle or spot representing a simple root α writing the value of λ_α which from Theorem 5 will always be a non-negative integer. From the λ_α the vector λ can be found in terms of the simple roots, and from λ the representation can be calculated. The following are examples of Schouten diagrams for various representations of A_2 and G_2:



etc.

Representations whose greatest weights cannot be obtained by adding multiples of the greatest weights of other representations are termed ' basic representations '. A representation is therefore basic if and only if λ_α ($\alpha \in \Pi$) is zero for all α except one. Non-basic representations may be derived from basic representations by a method described in Chapter 4, known as Cartan composition. Basic representations such that the simple root α for which $\lambda_\alpha \neq 0$ is perpendicular to all the other simple roots except one, are called ' elementary representations '. Some examples of elementary representations are shown below:



Basic representations can be derived from the two, or for some groups three elementary representations by a process known as alternation. In practice it is often easier to calculate a representation from first principles than to derive it from other representations.

3.2 Calculation of dimensions of irreducible representations

The dimension of a representation corresponding to a greatest weight λ depends directly on the vector λ in the following way:

Theorem 6 [11]

The dimension N of an irreducible representation corresponding to a greatest weight λ is given by

$$N = \prod_{\alpha\in\Sigma_+} \frac{(\lambda+g,\alpha)}{(g,\alpha)}. \tag{2}$$

In this equation Σ_+ is the set of all positive roots, and the vector g is half the sum of all the positive roots, i.e.

$$g = \tfrac{1}{2}\sum_{\alpha\in\Sigma_+} \alpha. \tag{3}$$

The vector g can also be found from its property

$$g_\alpha = \frac{2(g,\alpha)}{(\alpha,\alpha)} = 1. \tag{4}$$

It is sometimes more useful to use equation (4) and put equation (2) in the form

$$N = \prod_{\alpha\in\Sigma_+}\left[\frac{(\lambda,\alpha)}{(g,\alpha)}+1\right]$$

$$= \prod_{\alpha\in\Pi}(\lambda_\alpha+1)\prod_{\substack{\beta\in\Sigma_+\\ \beta\notin\Pi}}\left[\frac{(\lambda,\beta)}{(g,\beta)}+1\right]. \tag{5}$$

The following examples use the sets of roots calculated in Chapter 2 to calculate the dimensions of some of the smaller representations of groups B_1, A_2 and G_2.

Example 1

Consider the group B_1. Since there is only one positive root, the simple root α, the dimension of the representation can be calculated for a general case where the weight vector λ is such that $\lambda_\alpha = n$, n being a positive integer.

The Schouten diagram is:

giving

$$\lambda_\alpha = \frac{2(\lambda,\alpha)}{(\alpha,\alpha)} = n. \tag{6}$$

The vector g in equation (2) is given by

$$g = \tfrac{1}{2}\alpha \tag{7}$$

but in this case we use equation (4), i.e.

$$g_\alpha = \frac{2(g,\alpha)}{(\alpha,\alpha)} = 1 \tag{8}$$

so

$$N = \frac{\lambda_\alpha}{g_\alpha} + 1 = n+1. \tag{9}$$

It follows that there are representations of the group B_1—the group of operators forming rotations in three dimensions—in any number of dimensions. The representation in two dimensions is the well-known spinor representation and that in three dimensions corresponds to the vector product in three dimensions discussed in Chapter 1.

Example 2

We now turn to the next simplest algebra, the unitary algebra A_2. This is the algebra which has application in the theory of elementary particles. The dimensions of representations are particularly significant since the elementary particles are taken to be in one-to-one correspondence with the vectors spanning the representation spaces. It follows that the particles fall into multiplets of number equal to the dimension of some of the representations of this algebra.

This algebra can also be treated in the general case. Consider the Schouten diagram

Since

$$\Sigma_+ \equiv \{\alpha, \beta, \alpha+\beta\}, \tag{10}$$

$$\left.\begin{aligned} g_\alpha &= 1, \\ g_\beta &= 1, \end{aligned}\right\} \tag{11}$$

and

$$(\alpha,\alpha) = (\beta,\beta), \tag{12}$$

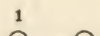
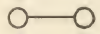
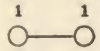
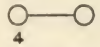
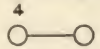
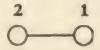
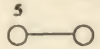
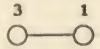
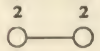
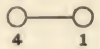
it follows that

$$N = (\lambda_\alpha+1)(\lambda_\beta+1)\left(\frac{\lambda_\alpha+\lambda_\beta}{2}+1\right)$$

$$= \tfrac{1}{2}(n+1)(m+1)(n+m+2). \tag{13}$$

From this formula the dimensions of all the representations of A_2 can easily be calculated. Those of dimension less than 50 are listed in Table 1.

Table 1

Representation	*Dimension*
	3
	6
	8
	10
	15
	15
	21
	24
	27
	28
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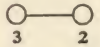
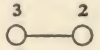
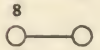
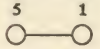
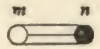
	36
	42
	45
	48

Example 3

Our third example is the simplest exceptional algebra, G_2. It was shown in Chapter 2 that its positive roots were

$$\Sigma_+ \equiv \{\alpha, \beta, \alpha+\beta, \alpha+2\beta, \alpha+3\beta, 2\alpha+3\beta\},$$

where α is the long simple root and β the short one. For the general representation



we therefore have

$$N = \prod_{\mu \in \Sigma_+} \left[\frac{(\lambda, \mu)}{(\mu, \mu)} + 1 \right]. \tag{14}$$

Since

$$\mu = \sum_{\alpha \in \Pi} a_\alpha^\mu \alpha \tag{15}$$

$$(\lambda, \mu) = \tfrac{1}{2} \sum_{\alpha \in \Pi} a_\alpha^\mu \lambda_\alpha (\alpha, \alpha)$$

$$= \tfrac{1}{2}\lambda[3a_\alpha^\mu m + a_\beta^\mu n] \tag{16}$$

$$(\lambda, g) = \tfrac{1}{2}\lambda[3a_\alpha^\mu + a_\beta^\mu] \tag{17}$$

and N becomes:

$$N = (m+1)(n+1)\left(\frac{3m+n}{4}+1\right)\left(\frac{3m+2n}{5}+1\right)\times$$
$$\times\left(\frac{3m+3n}{6}+1\right)\left(\frac{6m+3n}{9}+1\right)$$

$$= \frac{1}{4.5.6}(m+1)(n+1)(3m+n+4)(3m+2n+5)\times$$
$$\times(m+n+2)(2m+n+3). \tag{18}$$

This formula is complicated, and if the dimension of one particular representation only were required it would be easier to use equation (2) directly. Substituting various integers for m and n in equation (18) we can again construct a table of the representations of dimension less than 100, see Table 2.

Table 2

Representation	*Dimension*
	7
	14
	27
	64
	77
	77

3.3 Derivation of weight vectors from greatest weight

We have seen that any representation of a simple Lie algebra is completely specified by its greatest weight. From this greatest weight the dimension of the representation can be calculated from equation (2). From the greatest weight we can also calculate the ' weight vectors '. These are a set of vectors in the idempotent equal in number to the dimension n of the representation. It will be seen later that the number of different weight vectors will be less than the number of dimensions if degeneracy occurs, when one weight vector will correspond to two or more dimensions. To each weight vector λ_s there corresponds a vector spanning the representation space labelled ξ_s and called a basic vector of the representation. A general vector in the representation space is therefore a linear combination of the basic vectors and can be written as a column with n units, each unit being the amplitude of the corresponding basic vector. Thus a general representation vector may be written

$$\xi = \sum_{s=1}^{n} a_s^\xi \xi_s = \begin{pmatrix} a_1^\xi \\ a_2^\xi \\ \cdot \\ \cdot \\ a_n^\xi \end{pmatrix}. \tag{19}$$

The application of an operator of the Lie algebra to a representation vector, changes it into another representation vector, i.e.

$$E_\alpha \xi = \xi' \tag{20}$$

and the matrix elements of the $n \times n$ matrix in one-to-one correspondence with E_α in this representation depend on scalar products of the root α to which E_α is related and the weight vectors λ_s to which the elements of the vector ξ are related. The derivation of weight vectors from the greatest weight is therefore fundamental to the explicit calculation of the matrices forming a representation of a Lie algebra.

The following theorem gives a rule for deriving the set of weight vectors from the greatest weight. The similarity with the rule for deriving roots from simple roots will immediately be apparent.

Theorem 7 **[11]**

If λ_s is a weight vector ($\lambda_s \in \Delta$) then

$$\lambda_s - \alpha \in \Delta, \quad \alpha \in \Pi \tag{21}$$

if and only if

$$\frac{2(\lambda_s, \alpha)}{(\alpha, \alpha)} + Q(\lambda_s, \alpha) > 0, \tag{22}$$

where

$$\left.\begin{aligned} \lambda_s + Q(\lambda_s, \alpha)\alpha \in \Delta, \\ \text{while} \\ \lambda_s + [Q(\lambda_s, \alpha) + 1]\alpha \bar{\in} \Delta. \end{aligned}\right\} \tag{23}$$

Thus the procedure of calculating the set Δ is to start with the greatest weight λ and subtract simple roots from it one by one, labelling the results as belonging or not belonging to the set Δ according as condition (22) is or is not obeyed. The value of $Q(\lambda_s, \alpha)$ can then be discovered at any point by examination of the vectors already labelled as belonging to Δ.

Some examples of calculations of weight vectors will clarify this and be of use when calculating matrices in Chapter 4.

Example 4

We shall start with the simplest representation of the simplest algebra, i.e.

$$\overset{1}{\bigcirc}$$

where

$$\frac{2(\lambda, \alpha)}{(\alpha, \alpha)} = 1 \tag{24}$$

or

$$\lambda = \tfrac{1}{2}\alpha \tag{25}$$

taking

$$(\alpha, \alpha) = 2. \tag{26}$$

Applying Theorem 7

$$\lambda - \alpha = -\tfrac{1}{2}\alpha \in \Delta \tag{27}$$

if and only if

$$\frac{2(\lambda, \alpha)}{(\alpha, \alpha)} + Q(\lambda, \alpha) > 0 \tag{28}$$

and since

$$Q(\lambda, \alpha) = 0 \tag{29}$$

$$-\tfrac{1}{2}\alpha \in \Delta. \tag{30}$$

The weight system Δ therefore consists of $\frac{1}{2}\alpha$ and $-\frac{1}{2}\alpha$, i.e.

$$\Delta \equiv \{\tfrac{1}{2}\alpha, -\tfrac{1}{2}\alpha\} \tag{31}$$

confirming that the representation

$$\overset{1}{\bigcirc}$$

is two-dimensional.

Now consider

$$\overset{2}{\bigcirc}.$$

In this case

$$\lambda_1 = \alpha \tag{32}$$

and

$$\lambda_2 = \alpha - \alpha = 0 \in \Delta \tag{33}$$

since

$$\frac{2(\lambda_1, \alpha)}{(\alpha, \alpha)} + Q(\lambda_1, \alpha) > 0. \tag{34}$$

Similarly,

$$\lambda_3 = 0 - \alpha = -\alpha \in \Delta \tag{35}$$

since

$$\frac{2(\lambda_2, \alpha)}{(\alpha, \alpha)} + Q(\lambda_2, \alpha) > 0, \tag{36}$$

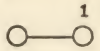
$Q(\lambda_2, \alpha)$ being unity. The weight system of $\overset{2}{\bigcirc}$ is therefore

$$\Delta \equiv \{\alpha, 0, -\alpha\} \tag{37}$$

and the representation is confirmed as being three-dimensional.

Example 5

For the algebra A_2 we shall calculate the weights of the lowest representation, namely



Its greatest weight is

$$\lambda_1 = \tfrac{1}{3}\alpha + \tfrac{2}{3}\beta \tag{38}$$

so, since

$$(\lambda_1)_\beta = \frac{2(\lambda_1, \beta)}{(\beta, \beta)} \tag{39}$$

is the only non-zero component,

$$\lambda_2 = \lambda_1 - \beta = \tfrac{1}{3}\alpha - \tfrac{1}{3}\beta \tag{40}$$

and since

$$\frac{2(\lambda_2, \alpha)}{(\alpha, \alpha)} + Q(\lambda_2, \alpha) > 0, \tag{41}$$

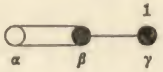
$$\lambda_3 = -(\tfrac{2}{3}\alpha + \tfrac{1}{3}\beta) \tag{42}$$

giving

$$\Delta \equiv \{\tfrac{1}{3}\alpha + \tfrac{2}{3}\beta, \tfrac{1}{3}\alpha - \tfrac{1}{3}\beta, -(\tfrac{2}{3}\alpha + \tfrac{1}{3}\beta)\}. \tag{43}$$

Example 6

As our next example we shall calculate the weight system of one of the three basic representations of C_3, namely



which is of dimension six. Its greatest weight is

$$\lambda_1 = \tfrac{1}{2}\alpha + \beta + \gamma. \tag{44}$$

Since $(\lambda_1)_\gamma$ is the only non-zero component of λ_1, it follows that the second weight is

$$\lambda_2 = \tfrac{1}{2}\alpha + \beta. \tag{45}$$

For the rest it is simplest to construct a table similar to the table used in Chapter 2 for finding roots. In the first column we put the weight vector under consideration, in the second the relevant scalar product term, in the third the value of Q and in the last column the vector is either labelled as a weight or marked as not being a member of the set Δ.

Table 3

Vector	*Scalar product term*	*Q-term*	*Conclusion*
$\frac{1}{2}\alpha+\beta-\gamma$	-1	1	No
$\frac{1}{2}\alpha$	1	0	$\lambda_3 \in \Delta$
$-\frac{1}{2}\alpha+\beta$	0	0	No
$-\frac{1}{2}\alpha$	1	0	$\lambda_4 \in \Delta$
$\frac{1}{2}\alpha-\beta$	-1	1	No
$\frac{1}{2}\alpha-\gamma$	0	0	No
$-\frac{1}{2}\alpha-\beta$	1	0	$\lambda_5 \in \Delta$
$-\frac{3}{2}\alpha$	-1	1	No
$-\frac{1}{2}\alpha-\gamma$	0	0	No
$-\frac{1}{2}\alpha-\beta-\gamma$	1	0	$\lambda_6 \in \Delta$
$-\frac{3}{2}\alpha-\beta$	0	0	No
$-\frac{1}{2}\alpha-2\beta$	-1	1	No

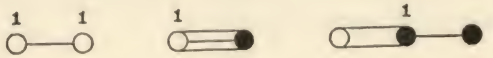
From this we conclude that

$$\Delta \equiv \{\tfrac{1}{2}\alpha+\beta+\gamma, \tfrac{1}{2}\alpha+\beta, \tfrac{1}{2}\alpha, -\tfrac{1}{2}\alpha, -\tfrac{1}{2}\alpha-\beta, -\tfrac{1}{2}\alpha-\beta-\gamma\}. \tag{46}$$

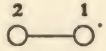
3.4 Properties of weight vectors

In this section the properties of the system of weight vectors Δ will be discussed.

One point is that it sometimes happens that the number of different weight vectors is less than the dimension of the representation. In this case there is a degeneracy in the representation space with two or more representation vectors corresponding to the same weight vector in the idempotent. This is a common occurrence with the zero weight vector; for example it occurs in the representations



but also happens sometimes with non-zero weight vectors, e.g. in the representation



When this occurs an arbitrariness is introduced into the representation, since the representation vectors spanning the degenerate part of the representation space can be chosen arbitrarily—the only restriction being that they must be orthogonal.

Another point is that weight vectors are vectors in the idempotent and can be expressed as linear combinations of simple roots, thus:

$$\lambda_s = \sum_{\alpha \in \Pi} a_\alpha^s \alpha. \tag{47}$$

It follows that the ' power ' of a weight vector can be defined in an analogous way to the power of a root (see Chapter 2, equations (22) and (23)), i.e.

$$\delta(\lambda_s) = \sum_{\alpha \in \Pi} a_\alpha^s. \tag{48}$$

Since weights are formed by subtracting the simple roots one by one, the powers of the weights will differ from the power of the greatest weight by integers. Thus

$$\gamma(\lambda_s) = \delta(\lambda_1) - \delta(\lambda_s) \tag{49}$$

gives the number of simple roots subtracted from λ_1 to obtain λ_s. Any set of weight vectors for which $\gamma(\lambda_s)$ is the same, k say, is called a ' layer ' and written Δ^k. The number of weight vectors in the layer Δ^k is written s_k. The width of the representation is the value of the greatest s_k and the height of the representation T is given by

$$T = 2\delta(\lambda_1) + 1. \tag{50}$$

This terminology suggests the construction of weight diagrams where the weight vectors are arranged in layers, the greatest at the top and the other weights arranged below in layers of decreasing power.

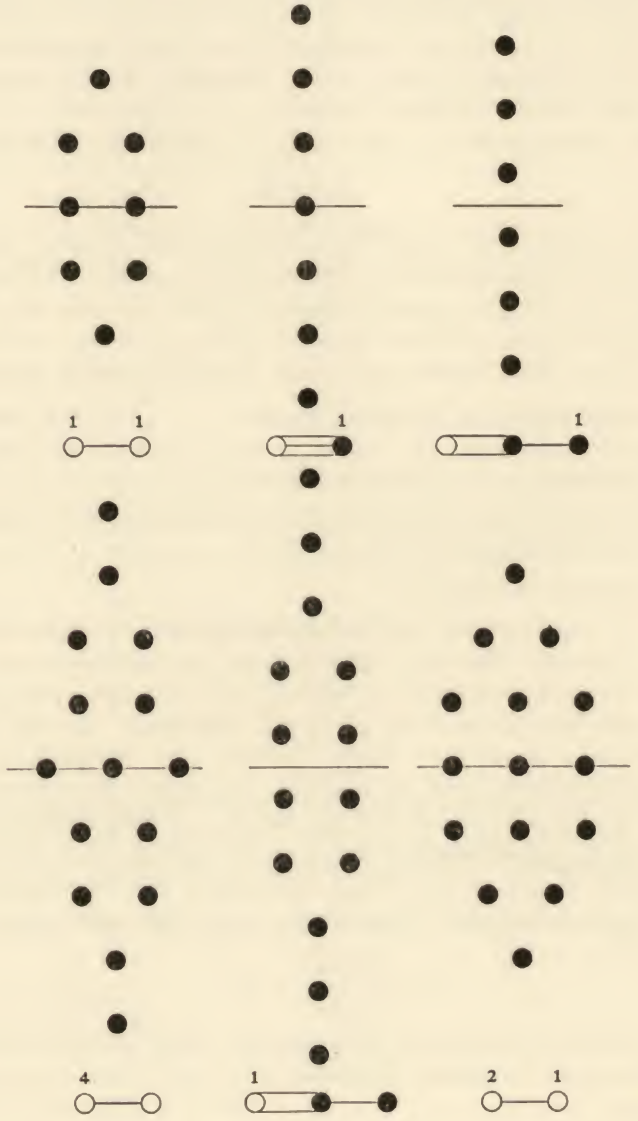
Theorem 8 **[11]**

If the system of weights of an irreducible representation of a simple Lie algebra is arranged according to the power of the weights, it forms a symmetrical spindle shape, i.e.

(i) $$s_1 \leqslant s_2 \leqslant \ldots \leqslant s_r \qquad \left(r = \frac{T}{2},\ T \text{ even};\quad r = \frac{T+1}{2},\ T \text{ odd}\right). \tag{51}$$

(ii) $$s_k = s_{T-k+1}. \tag{52}$$

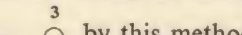
The first part of the theorem ensures a spindle shape, the second ensures the symmetry. The symmetry is a reflection about the $\delta(\lambda) = 0$ line, and, since the powers of weights differ by integers,



determines that these powers must either be integral or half-integral. The height T may then be even or odd—if T is odd the powers of the weights are integers, the width of the representation is the number of weights of zero power and the oddness or evenness of the dimension of the representation varies with the oddness or evenness of its width. If T is even, the powers of the weights are half-integral and the dimension is always even. Examples of weight diagrams are shown in the previous diagram.

The symmetry of the weight diagram together with the fact that

$$\sum_{\mu\in\Delta} \mu = 0 \tag{53}$$

(which follows from the fact that the traces of all diagonal matrices are zero, see also Chapter 4) often makes it possible to write down most of the weight vectors of negative power once knowing those of positive power. Caution should be taken, however, as mistakes may arise. (The reader may amuse himself by trying to deduce the weight vectors of the representation  by this method.)

Following Theorem 8 it is often possible to draw weight diagrams from knowledge of the representation diagram, the power of the greatest weight, and the dimension of the representation. Examples of this will be found at the end of 5.3.

3.5 Summary of Chapter 3

1. The operators of a simple Lie algebra can be represented by a set of matrices. The only restriction on these matrices is that they obey the same commutation relations as the operators; the result is that such sets can be found of various dimensions. To each representation corresponds a vector known as the 'greatest weight' which is a vector in the same space as the roots. Since the simple roots span this space, the greatest weight λ is fully determined by its scalar products with the simple roots, and therefore the representation of any chosen simple Lie algebra can be classified by drawing the Schouten diagram and above each spot writing the value of

$$\lambda_\alpha = \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}, \quad \alpha\in\Pi.$$

For a vector in this space to be the greatest weight of a representation of the simple Lie algebra the values of λ_α must all be non-negative integers.

2. The dimension N of the representation is given by

$$N = \prod_{\alpha\in\Sigma_+} \frac{(\lambda+g,\alpha)}{(g,\alpha)}$$

where Σ_+ is the set of all positive roots and the vector is defined by

$$g = \tfrac{1}{2}\sum_{\alpha\in\Sigma_+} \alpha.$$

3. The set of weight vectors Δ is the set of those vectors satisfying the following condition:

If

$$\lambda'\in\Delta$$
$$\lambda'-\alpha\in\Delta, \quad \alpha\in\Pi,$$

if and only if

$$\frac{2(\lambda',\alpha)}{(\alpha,\alpha)} + Q(\lambda',\alpha) > 0,$$

where $Q(\lambda',\alpha)$ is such that

$$\lambda'+Q(\lambda',\alpha)\alpha\in\Delta$$

but

$$\lambda'+[Q(\lambda',\alpha)+1]\alpha\bar{\in}\Delta.$$

The total number of weight vectors is equal to the dimension of the representation. There may be more than one weight vector with any value.

4. The power of a weight vector is defined by

$$\delta(\lambda_s) = \sum_{\alpha\in\Pi} a_\alpha^s$$

where

$$\lambda_s = \sum_{\alpha\in\Pi} a_\alpha^s \alpha.$$

The number of weight vectors in a set Δ which have the same power k is labelled s_k. If the system of weights Δ is arranged according to the power of its weights it forms a symmetrical spindle shape, i.e.

$$s_1 \leqslant s_2 \leqslant \ldots \leqslant s_r \quad \left(r = \frac{T}{2},\ T \text{ even}; \quad r = \frac{T+1}{2},\ T \text{ odd},\right)$$

$$s_k = s_{T-k+1}$$

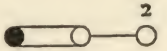
where T is given by

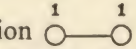
$$T = 2\delta(\lambda_1)+1$$

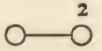
and λ_1 is the greatest weight.

PROBLEMS 3

1. Find the dimension of the representation

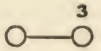


2. Find which representations of A_3 are of dimension less than 70.
3. Find the general formula for the dimensions of representations of the algebra B_2.
4. Calculate the weight vectors of the representation  and draw its weight diagram. Why is the number of weight vectors not equal to the dimension of the representation? Draw the weight vectors in the two-dimensional idempotent.
5. Calculate the weight vectors of



and draw its weight diagram.

6. Show that the greatest weight of the representation



is

$$\alpha+2\beta.$$

7. Show that the greatest weight of the representation

is

$$2\alpha+3\beta.$$

CHAPTER 4

Calculation of the Matrix Elements of Representations from First Principles

In this and the following two chapters we shall describe the various methods of calculating explicitly the matrix elements of the matrices which represent the operators of simple Lie algebras. There are three methods—direct calculation from the weight vectors; Kronecker multiplication of known representations followed by reduction; alternation followed by reduction if necessary. The first method of direct calculation is the simplest unless there is a degeneracy† among the weights. It should therefore be used for all representations where there is no degeneracy and indeed is the only method available for elementary representations.

4.1 Calculation of matrix elements from weight vectors

The calculation follows three stages:

(i) Calculation of the matrix elements of the diagonal matrices which correspond to operators in the idempotent following Theorem 9.

(ii) Specification of which elements of the non-diagonal matrices corresponding to operators not in the idempotent are zero and which are not using Theorem 10.

(iii) Calculation of the values of the non-zero elements of these matrices. Theorem 11 is sufficient to determine the numerical value of these for representations with non-diagonal weight systems.

Theorem 9 **[11]**

For diagonal operators

$$H_\sigma \xi_{\lambda_s} = (\sigma, \lambda_s)\xi_{\lambda_s}. \tag{1}$$

† See 3.4.

where (σ, λ_s) is the scalar product in the idempotent of the vectors σ ($\in \Gamma$) and λ_s ($\in \Delta$) to which the operator H_σ and the basic vector $\xi\lambda_s$ correspond.

Both σ and λ_s are normally expressed in terms of the simple roots of the algebra whose scalar products are known explicitly from examination of the Schouten diagram.†

The remainder of the calculation follows directly from the commutation relations

$$[F_\mu, E_\alpha] = -(\mu, \alpha)E_\alpha \tag{2}$$

$$[E_\alpha, E_{-\alpha}] = F_\alpha = \sum_{\mu \in \Gamma} a_\alpha^\mu F_\mu \tag{3}$$

$$[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta} \tag{4}$$

$$\text{for } \alpha+\beta \in \Sigma,\ \mu \in \Gamma,\ \alpha, \beta \in \Sigma.$$

Theorem 10 follows from the first, and Theorem 11 from the second. The third is necessary to find the numerical values of the matrix elements when there is a degeneracy in the weight system, and puts restrictions on the relative signs of the matrix elements.

Theorem 10

All the elements $E^r_{\alpha s}$ of E_α ($\alpha \in \Sigma$) are zero except those for which

$$\lambda_s - \lambda_r = \alpha. \tag{5}$$

Proof:

The commutation relation (2) can be written

$$F_{\mu\ r}^{\ r} E_{\alpha\ s}^{\ r} - E_{\alpha\ s}^{\ r} F_{\mu\ s}^{\ s} = -(\mu, \alpha) E_{\alpha\ s}^{\ r} \tag{6}$$

from which either

$$E_{\alpha\ s}^{\ r} = 0 \tag{7}$$

or

$$F_{\mu\ r}^{\ r} - F_{\mu\ s}^{\ s} = -(\mu, \alpha). \tag{8}$$

From Theorem 9 we can write equation (8) as

$$(\mu, \lambda_r) - (\mu, \lambda_s) = -(\mu, \alpha) \tag{9}$$

or

$$(\mu, \lambda_r - \lambda_s + \alpha) = 0 \tag{10}$$

† See 2.2.

from which it follows that either

$$\lambda_r - \lambda_s + \alpha = 0 \tag{11}$$

or $\lambda_r - \lambda_s + \alpha$ is a vector orthogonal to μ in the idempotent, and must therefore be a member of the set Γ. Since μ has been chosen arbitrarily, a relation like equation (10) holds for μ replaced by any other member of Γ and we conclude that, if

$$E_{\alpha\ s}^{\ r} \neq 0 \tag{12}$$

$$\alpha = \lambda_s - \lambda_r. \tag{13}$$

Theorem 11

The numerical value of the matrix element $E_{\alpha\ r}^{\ t}$ is given by

$$E_{\alpha\ r}^{\ t} = \pm\sqrt{F_{\alpha\ r}^{\ r} + (E_{\alpha\ s}^{\ r})^2} \tag{14}$$

if there is no degeneracy in the weight system.

Proof:

The commutation relation (3) can be written

$$E_{\alpha\ s}^{\ r} E_{-\alpha\ r}^{\ s} - E_{-\alpha\ t}^{\ r} E_{\alpha\ r}^{\ t} = F_{\alpha\ r}^{\ r} \tag{15}$$

if there is no degeneracy. If there were degeneracy the terms on the left-hand side would be summed over s and t respectively. Since

$$E_{-\alpha\ r}^{\ s} = -E_{\alpha\ s}^{\ r} \tag{16}$$

(see [**10**]) equation (15) can be written

$$-(E_{\alpha\ s}^{\ r})^2 + (E_{\alpha\ r}^{\ t})^2 = F_{\alpha\ r}^{\ r} \tag{17}$$

from which Theorem 11 immediately follows.

It is understood when writing equation (15) that either term on the left-hand side may be zero, i.e. there may be no λ_s such that for any chosen λ_r and α

$$\lambda_s - \lambda_r = \alpha. \tag{18}$$

(So long as we restrict ourselves to weight systems without degeneracy there will not be two such weights λ_s.) If λ_r is taken to be either the greatest weight or the least weight, then there will be only one term on the left-hand side of equation (15) and equation (17) for all α's. In practice we start with λ_r the least vector and work up. For the

first application of Theorem 11 the term $(E_\alpha^r{}_s)^2$ on the right-hand side of equation (14) is then zero. For all subsequent applications it is either zero or already known.

The signs of these matrix elements, whose numerical values follow from Theorem 11, are restricted by the condition that the matrices obey equation (4). Generally there are a great many possible sets of signs.

If there is a degeneracy in the weight system, the revised form of Theorem 11 does not give a sufficient number of equations to determine the matrix elements. It is necessary to use the set of equations given by equation (4) as well. The resulting set of simultaneous equations can be extremely complicated to deal with. The first step is to choose a basis in the degenerate space by arbitrarily choosing the values of the relevant matrix elements of one of the operators. The examples below will clarify the procedure.

Example 1

The simplest representation of a Lie algebra is the spinor representation of the algebra B_1, namely

Taking α as the simple root, the weight vectors are†

$$\Delta \equiv \{\tfrac{1}{2}\alpha, -\tfrac{1}{2}\alpha\} \tag{19}$$

and using Theorem 9 the diagonal matrix F_α has elements given by

$$F_\alpha \xi_{\frac{1}{2}\alpha} = (\tfrac{1}{2}\alpha, \alpha)\xi_{\frac{1}{2}\alpha} = \xi_{\frac{1}{2}\alpha} \tag{20}$$

$$F_\alpha \xi_{-\frac{1}{2}\alpha} = (-\tfrac{1}{2}\alpha, \alpha)\xi_{-\frac{1}{2}\alpha} = -\xi_{-\frac{1}{2}\alpha} \tag{21}$$

so

$$\mathrm{F}_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{22}$$

where

$$\left.\begin{aligned} \xi_{\frac{1}{2}\alpha} &= \xi_1 \\ \xi_{-\frac{1}{2}\alpha} &= \xi_2. \end{aligned}\right\} \tag{23}$$

† See 3.3, Example 4.

Applying Theorem 10, the only non-zero element of E_α is $E_\alpha^2{}_1$, and from Theorem 11 we have

$$(E_\alpha^2{}_1)^2 = F_\alpha^1{}_1 = 1 \tag{24}$$

so

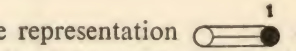
$$E_\alpha^2{}_1 = \pm 1 \tag{25}$$

and

$$E_\alpha = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad E_{-\alpha} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}. \tag{26}$$

The three matrices in equations (22) and (26) or linear combinations of them form the well-known spinor representation.

Example 2

For this example we shall examine the representation  of the algebra G_2. This is a seven-dimensional representation having for weight vectors the set

$$\Delta \equiv \{\alpha+2\beta, \alpha+\beta, \beta, 0, -\beta, -(\alpha+\beta), -(\alpha+2\beta)\}. \tag{27}$$

From Theorem 9 the diagonal operators can be written immediately

$$F_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ & & & 0 & 0 & 0 & 0 \\ & & & & 1 & 0 & 0 \\ & & & & & -1 & 0 \\ \text{etc.} & & & & & & 0 \end{pmatrix}$$

$$F_\beta = \begin{pmatrix} \frac{1}{3} & & & & & & \\ & -\frac{1}{3} & & & & & \\ & & \frac{2}{3} & & & & \\ & & & 0 & & & \\ & & & & -\frac{2}{3} & & \\ & & & & & \frac{1}{3} & \\ & & & & & & -\frac{1}{3} \end{pmatrix}. \tag{28}$$

It is convenient when Theorem 11 is used to have the diagonal matrices corresponding to the other roots to refer to. These are:

$$F_{\alpha+\beta} = F_\alpha + F_\beta = \begin{pmatrix} \frac{1}{3} & & & & & & \\ & \frac{2}{3} & & & & & \\ & & -\frac{1}{3} & & & & \\ & & & 0 & & & \\ & & & & \frac{1}{3} & & \\ & & & & & -\frac{2}{3} & \\ & & & & & & -\frac{1}{3} \end{pmatrix}$$

$$F_{\alpha+2\beta} = F_{\alpha+\beta} + F_\beta = \begin{pmatrix} \frac{2}{3} & & & & & & \\ & \frac{1}{3} & & & & & \\ & & \frac{1}{3} & & & & \\ & & & 0 & & & \\ & & & & -\frac{1}{3} & & \\ & & & & & -\frac{1}{3} & \\ & & & & & & -\frac{2}{3} \end{pmatrix} \tag{29}$$

$$F_{\alpha+3\beta} = F_{\alpha+2\beta} + F_\beta = \begin{pmatrix} 1 & & & & & & \\ & 0 & & & & & \\ & & 1 & & & & \\ & & & 0 & & & \\ & & & & -1 & & \\ & & & & & 0 & \\ & & & & & & -1 \end{pmatrix}$$

$$F_{2\alpha+3\beta} = F_{\alpha+2\beta} + F_{\alpha+\beta} = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 0 & & & & \\ & & & 0 & & & \\ & & & & 0 & & \\ & & & & & -1 & \\ & & & & & & -1 \end{pmatrix}.$$

The non-zero elements of E_α, for instance, can be seen by examination of the set Δ in equation (27). If there is any weight ξ_s formed by adding α to a weight ξ_t, then the element $E_\alpha{}^s{}_t$ is non-zero. Thus

$$E_\alpha{}^3{}_2 \neq 0, \qquad E_\alpha{}^6{}_5 \neq 0. \tag{30}$$

From Theorem 11

$$(E_\alpha{}^6{}_5)^2 = -F_\alpha{}^6{}_6 = F_\alpha{}^5{}_5 = 1 \tag{31}$$

so

$$E_\alpha{}^6{}_5 = \pm 1. \tag{32}$$

Similarly

$$E_\alpha{}^3{}_2 = \pm 1. \tag{33}$$

For E_β the non-zero elements and their values can be written down directly from examination of equations (27) and (28).

$$\left.\begin{aligned} E_\beta{}^7{}_6 &= \pm\sqrt{\tfrac{1}{3}}, \\ E_\beta{}^5{}_4 &= \pm\sqrt{\tfrac{2}{3}}, \\ E_\beta{}^4{}_3 &= \pm\sqrt{\tfrac{2}{3}}, \\ E_\beta{}^2{}_1 &= \pm\sqrt{\tfrac{1}{3}}. \end{aligned}\right\} \tag{34}$$

Note that Theorem 11 gives the equations

$$\left.\begin{aligned} -(E_\beta{}^5{}_4)^2 &= E_\beta{}^5{}_5, \\ (E_\beta{}^5{}_4)^2 - (E_\beta{}^4{}_3)^2 &= F_\beta{}^4{}_4, \\ (-E_\beta{}^4{}_3)^2 &= F_\beta{}^3{}_3, \end{aligned}\right\} \tag{35}$$

so having discovered the value of $E_\beta{}^5{}_4$ from the first of these, the zero value of $F_\beta{}^4{}_4$ implies that there is a non-zero element of E_β with superscript 4.

The remaining matrices are computed in a similar manner leading to:

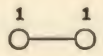
$$\left.\begin{aligned} E_{\alpha+\beta}{}^7{}_5 &= \pm\sqrt{\tfrac{1}{3}}, & E_{\alpha+2\beta}{}^7{}_4 &= \pm\sqrt{\tfrac{2}{3}}, \\ E_{\alpha+\beta}{}^6{}_4 &= \pm\sqrt{\tfrac{2}{3}}, & E_{\alpha+2\beta}{}^6{}_3 &= \pm\sqrt{\tfrac{1}{3}}, \\ E_{\alpha+\beta}{}^4{}_2 &= \pm\sqrt{\tfrac{2}{3}}, & E_{\alpha+2\beta}{}^5{}_2 &= \pm\sqrt{\tfrac{1}{3}}, \\ E_{\alpha+\beta}{}^3{}_1 &= \pm\sqrt{\tfrac{1}{3}}, & E_{\alpha+2\beta}{}^4{}_1 &= \pm\sqrt{\tfrac{2}{3}}, \\ \\ E_{\alpha+3\beta}{}^7{}_3 &= \pm 1, & E_{2\alpha+3\beta}{}^7{}_2 &= \pm 1, \\ E_{\alpha+3\beta}{}^5{}_1 &= \pm 1, & E_{2\alpha+3\beta}{}^6{}_1 &= \pm 1. \end{aligned}\right\} \tag{36}$$

The calculation is seen to be simple though tedious. Restrictions on the signs follows from commutation relations of the form

$$[E_\sigma, E_\rho] = N_{\sigma,\rho} E_{\sigma+\rho} \quad \sigma, \rho \in \Sigma.$$

Example 3

As a third example we shall calculate the representation



by this method, both because of its physical significance, and because it is a simple example of a representation with a degeneracy in its weight system. The set of weights is

$$\Delta \equiv \{\alpha+\beta, \alpha, \beta, 0, 0, -\beta, -\alpha, -(\alpha+\beta)\} \tag{37}$$

with a twofold degeneracy in the zero element. The diagonal matrices can be written directly as

$$F_\alpha = \begin{pmatrix} 1 & & & & & & & \\ & 2 & & & & & & \\ & & -1 & & & & & \\ & & & 0 & & & & \\ & & & & 0 & & & \\ & & & & & 1 & & \\ & & & & & & -2 & \\ & & & & & & & -1 \end{pmatrix}$$

$$F_\beta = \begin{pmatrix} 1 & & & & & & & \\ & -1 & & & & & & \\ & & 2 & & & & & \\ & & & 0 & & & & \\ & & & & 0 & & & \\ & & & & & -2 & & \\ & & & & & & 1 & \\ & & & & & & & -1 \end{pmatrix} \tag{38}$$

$$F_{\alpha+\beta} = F_\alpha + F_\beta = \begin{pmatrix} 2 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 0 & & & & \\ & & & & 0 & & & \\ & & & & & -1 & & \\ & & & & & & -1 & \\ & & & & & & & -2 \end{pmatrix}.$$

Expanding the commutation relation in equation (3) for each we find

$$\left.\begin{array}{ll} E_\alpha{}^3{}_1 = \pm 1, & E_\beta{}^2{}_1 = \pm 1, \\ (E_\alpha{}^4{}_2)^2 + (E_\alpha{}^5{}_2)^2 = 2, & (E_\beta{}^4{}_3)^2 + (E_\beta{}^5{}_3)^2 = 2, \\ (E_\alpha{}^7{}_4)^2 + (E_\alpha{}^7{}_5)^2 = 2, & (E_\beta{}^6{}_5)^2 + (E_\beta{}^6{}_4)^2 = 2, \\ E_\alpha{}^7{}_4 = \pm E_\alpha{}^4{}_2, & E_\beta{}^6{}_4 = \pm E_\beta{}^4{}_3, \\ E_\alpha{}^7{}_5 = \pm E_\alpha{}^5{}_2, & E_\beta{}^6{}_5 = \pm E_\beta{}^5{}_3, \\ E_\alpha{}^8{}_6 = \pm 1, & E_\beta{}^8{}_7 = \pm 1, \end{array}\right\}$$

$$\left.\begin{array}{c} (E_{\alpha+\beta}{}^4{}_1)^2 + (E_{\alpha+\beta}{}^5{}_1)^2 = 2, \\ E_{\alpha+\beta}{}^6{}_2 = \pm 1, \\ E_{\alpha+\beta}{}^7{}_3 = \pm 1, \\ (E_{\alpha+\beta}{}^8{}_4)^2 + (E_{\alpha+\beta}{}^8{}_5)^2 = 2, \\ E_{\alpha+\beta}{}^8{}_4 = \pm E_{\alpha+\beta}{}^4{}_1, \\ E_{\alpha+\beta}{}^8{}_5 = \pm E_{\alpha+\beta}{}^5{}_1. \end{array}\right\} \tag{39}$$

These equations replace Theorem 11 which cannot be used when there is a degeneracy in the weight system. They are not all independent, and there are not enough to determine all the unknowns. Use must be made of the commutation relation in equation (4), which becomes, after substitution of the elements determined by equation (39),

$$\left.\begin{array}{l} (\pm E_\alpha{}^4{}_2) - (\pm E_\beta{}^4{}_3) = E_{\alpha+\beta}{}^4{}_1, \\ (\pm E_\alpha{}^5{}_2) - (\pm E_\beta{}^5{}_3) = E_{\alpha+\beta}{}^5{}_1, \\ E_\beta{}^4{}_3 E_\alpha{}^4{}_2 \pm E_\beta{}^5{}_3 E_\alpha{}^5{}_2 = \pm 1, \end{array}\right\} \tag{40}$$

assuming

$$N_{\alpha,\beta} = +1. \tag{41}$$

A basis is chosen in the two-dimensional degenerate space of the weight system by arbitrarily taking

$$E_{\alpha+\beta}{}^4{}_1 = 0. \tag{42}$$

From equation (11) this gives

$$E_\alpha{}^4{}_2 = \pm E_\beta{}^4{}_3 = \pm a \tag{43}$$

say, and

$$E_{\alpha+\beta}{}^5{}_1 = \pm\sqrt{2} \tag{44}$$

from equations (39). Substituting equation (43) in equations (39) we have

$$E_\beta{}^5{}_3 = \pm E_\alpha{}^5{}_2 = \pm b \tag{45}$$

say, and substituting equation (45) in the second equation (40) we have

$$2b = \pm\sqrt{2} \tag{46}$$

giving

$$b = \pm \frac{1}{\sqrt{2}}. \tag{47}$$

Then from equations (39)

$$a^2 + b^2 = 2 \tag{48}$$

so

$$a = \pm\sqrt{\tfrac{3}{2}}. \tag{49}$$

The numerical values of all the matrix elements are then known, but it is seen that these values depend on the basis chosen in the degenerate space of the weight systems. The signs must be chosen so that the commutation relation in equation (4) is obeyed.

4.2 Summary of Chapter 4

Matrix elements can be calculated in the following way: a set of orthogonal vectors ξ_r ($r = 1, 2, \ldots, t$) in one-to-one correspondence with the weight vectors are taken to span the t-dimensional representation space. The action of the operators of the algebra on these vectors is determined numerically from relations between the weight vectors and the roots to which the operators are related.

The operators F_α are diagonal with elements given by

$$F_\alpha \xi_r = (\alpha, \lambda_r)\xi_r$$

where λ_r is the weight vector to which ξ_r corresponds.

The null elements of the operators E_μ ($\mu \in \Sigma$) are determined by

$$E_\mu \xi_r \propto \xi_s, \qquad \lambda_r + \mu = \lambda_s \in \Delta,$$
$$E_\mu \xi_r = 0, \qquad \lambda_r + \mu \bar{\in} \Delta.$$

The values of the non-zero elements are determined to within the sign by the relations

$$[E_\mu, E_{-\mu}] = \sum_{\alpha \in \Pi} a_\alpha^\mu F_\alpha,$$
$$\mu = \sum_{\alpha \in \Pi} a_\alpha^\mu \alpha,$$

when there is no degeneracy in the weight vectors. The relations

$$[E_\mu, E_\nu] = N_{\mu,\nu} E_{\mu+\nu}$$

restrict the signs. In case of degeneracy both relations must be used to determine the values of the matrix elements.

CHAPTER 5

Calculation of Matrix Elements of Representations from Basic Representations by Kronecker Multiplication

5.1 Introduction

The Kronecker product of two representations of a simple Lie algebra corresponds to the tensor product of the corresponding representations of the simple Lie group.

When the ' Kronecker product ' is taken of two representations of a simple Lie algebra, the result is also a representation of the algebra, though not necessarily irreducible. The basic vectors ζ_i of the final representation Φ are in one-to-one correspondence with the product of basic vectors of the original representations, the product always consisting of one basic vector from each representation, i.e.

$$\zeta_i = \xi_r \times \eta_s, \tag{1}$$

where the ξ_r ($r = 1, 2, \ldots, n$) are the basic vectors of the representation ϕ and the η_s ($s = 1, 2, \ldots, m$) those of the representation ψ. The Kronecker product is written

$$\Phi = \phi \times \psi. \tag{2}$$

From this definition it follows that the total number of basic vectors ζ_i in the representation Φ is nm. In other words, the dimension of the Kronecker product of two representations is equal to the product of the dimensions of the representations. In general, a simple Lie algebra having irreducible representations of dimensions n and m does not necessarily have an irreducible representation of dimension nm, and the result of taking the Kronecker product of two irreducible representations is a reducible representation—a direct sum of several irreducible representations.

The irreducible representation in the direct sum whose greatest weight is of highest power is called the ' greatest component ' of the Kronecker product. This is generally, but not always, also the irreducible representation of greatest dimension. (An example for which this is not the case will be found at the end of 5.3.) The greatest weight of the greatest component is the sum of the greatest weights of the two original representations, and the basic vector to which this greatest weight corresponds is

$$\zeta_1 = \xi_1 \times \eta_1, \tag{3}$$

where ξ_1 and η_1 are the basic vectors corresponding to the greatest weights of the original representations.

Usually the Kronecker product of two representations is taken in order to discover explicitly the greatest component of the resulting representation. The operation of Kronecker multiplication, together with the operation of separating the greatest component, is called the Cartan composition of the greatest component.

Below, the definition of the matrix elements of the Kronecker product of two representations is given in Theorem 12. The method of separating the greatest component is then described. Finally, the method of analysing the Kronecker product of two representations in order to find which representations occur in the direct sum, and to find any one explicitly is discussed.

5.2 Cartan composition

Theorem 12 **[11]**

If Φ is the Kronecker product of the representations ϕ and ψ, then the matrix representing the operator T_a in the representation Φ is given by

$$\Phi(T_a)\zeta_i = [\phi(T_a)\xi_r] \times \eta_s + \xi_r \times [\psi(T_a)\eta_s], \tag{4}$$

where $\phi(T_a)$ is the matrix representing the operator T_a in the representation ϕ with basic vectors ξ_t $(t = 1, 2, \ldots, n)$ and $\psi(T_a)$ is the matrix representing T_a in the representation ψ having basic vectors η_s $(s = 1, 2, \ldots, m)$. Since $\phi(T_a)$ and $\psi(T_a)$ are assumed to be known explicitly, we will have equations such as

$$\phi(T_a)\xi_r = a\xi_p \tag{5}$$

$$\psi(T_a)\eta_s = b\eta_t \tag{6}$$

giving

$$\begin{aligned} \Phi(T_a)\zeta_i &= a\xi_p \times \eta_s + b\xi_r \times \eta_t, \\ &= a\zeta_j + b\zeta_k, \end{aligned} \tag{7}$$

supposing

$$\left.\begin{aligned} \zeta_j &= \xi_p \times \eta_s, \\ \zeta_k &= \xi_r \times \eta_t. \end{aligned}\right\} \tag{8}$$

Thus the matrix elements of the matrix $\Phi(T_a)$ are found explicitly. The most significant step towards finding explicitly the greatest component of a Kronecker product, is finding which linear combinations of the basic vectors

$$\zeta_i = \xi_r \times \eta_s \tag{9}$$

are the ones which correspond to the weight vectors of the Kronecker product. This is done by starting with the greatest basic vector,† namely

$$\zeta_1 = \xi_1 \times \eta_1 \tag{10}$$

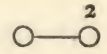
where ξ_1 and η_1 are the greatest basic vectors of the two original representations. Equation (4) is applied to this vector for all the non-diagonal operators. In each case, assuming there is no degeneracy, the result will either be zero, or it will be proportional to another basic vector of the greatest component. The process is then applied to all the basic vectors discovered by this method. Eventually, all the basic vectors and all the matrix elements will emerge.

If there is a degeneracy in the weight vectors, the process described above will in some cases lead to a linear combination of the basic vectors belonging to the greatest component. This will be discovered when it is seen that the process leads to basic vectors which are not orthogonal. In this case a set of orthogonal basic vectors must be chosen. This cannot always be chosen arbitrarily as sometimes one or more elements of the degenerate space is a basic vector belonging to another irreducible representation in the direct sum.

Some examples will clarify these points.

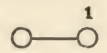
Example 1

For our first example we choose a representation with no degeneracy, namely



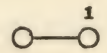
† The greatest basic vector is that corresponding to the greatest weight.

which can be calculated by Cartan composition, multiplying



by itself and then finding the greatest component.

The representation



calculated by the method of Chapter 4 has, for non-zero matrix elements of non-diagonal operators

$$\left.\begin{aligned} E_\alpha{}^3{}_2 &= \pm 1, \\ E_\beta{}^2{}_1 &= \pm 1, \\ E_{\alpha+\beta}{}^3{}_1 &= \pm 1, \end{aligned}\right\} \tag{11}$$

and

$$[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}. \tag{12}$$

Equation (12) restricts the signs in equation (11), but $N_{\alpha,\beta}$ can be chosen to be either plus or minus unity. Thus there are eight possible arrangements of signs in equation (11). The diagonal operators are represented by

$$\begin{aligned} F_\alpha &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\ F_\beta &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{13}$$

Taking

$$N_{\alpha,\beta} = +1 \tag{14}$$

and, for the two representations

$$\left.\begin{aligned} E_\alpha \xi_2 &= \xi_3, \\ E_\beta \xi_1 &= \xi_2, \\ E_{\alpha+\beta} \xi_1 &= \xi_3, \end{aligned}\right\} \tag{15}$$

and

$$\left.\begin{aligned} E_\alpha \eta_2 &= \eta_3, \\ E_\beta \eta_1 &= \eta_2, \\ E_{\alpha+\beta} \eta_1 &= \eta_3, \end{aligned}\right\} \tag{16}$$

we find the basic vectors involved in the greatest component by calculating the non-diagonal elements first, starting with the basic vector

$$\zeta_1 = \xi_1 \times \eta_1. \tag{17}$$

Applying each non-diagonal operator to this vector using Theorem 12 we have

$$\left.\begin{aligned} E_\alpha \xi_1 \eta_1 &= 0, \\ E_\beta \xi_1 \eta_1 &= \xi_2 \eta_1 + \xi_1 \eta_2 = \sqrt{2}\zeta_2, \\ E_{\alpha+\beta} \xi_1 \eta_1 &= \xi_3 \eta_1 + \xi_1 \eta_3 = \sqrt{2}\zeta_3, \end{aligned}\right\} \tag{18}$$

where

$$\left.\begin{aligned} \zeta_2 &= \frac{1}{\sqrt{2}}(\xi_2 \eta_1 + \xi_1 \eta_2), \\ \zeta_3 &= \frac{1}{\sqrt{2}}(\xi_3 \eta_1 + \xi_1 \eta_3), \end{aligned}\right\} \tag{19}$$

and two non-zero matrix elements are found, namely

$$\left.\begin{aligned} E_\beta{}^2{}_1 &= \sqrt{2}, \\ E_{\alpha+\beta}{}^3{}_1 &= \sqrt{2}. \end{aligned}\right\} \tag{20}$$

The factor $\sqrt{2}$ is inserted in equations (19) in order that the basic vectors normalise to unity. Proceeding:

$$\left.\begin{aligned} E_\alpha \zeta_2 &= \frac{1}{\sqrt{2}}(\xi_3 \eta_1 + \xi_1 \eta_3) = \zeta_3, \\ E_\beta \zeta_2 &= \sqrt{2}\xi_2 \eta_2 = \sqrt{2}\zeta_4, \\ E_{\alpha+\beta} \zeta_2 &= \frac{1}{\sqrt{2}}(\xi_2 \eta_3 + \xi_3 \eta_2) = \zeta_5, \end{aligned}\right\} \tag{21}$$

identifying

$$\left.\begin{aligned} \zeta_4 &= \xi_2 \eta_2, \\ \zeta_5 &= \frac{1}{\sqrt{2}}(\xi_2 \eta_3 + \xi_3 \eta_2). \end{aligned}\right\} \tag{22}$$

Continuing this method, we find

$$\zeta_6 = \xi_3 \eta_3 \tag{23}$$

and for non-zero matrix elements

$$\left.\begin{aligned} E_\alpha\zeta_4 &= \sqrt{2}\zeta_5 \\ E_\alpha\zeta_5 &= \sqrt{2}\zeta_6 \end{aligned}\right\} \tag{24}$$

$$E_\beta\zeta_3 = \zeta_5 \tag{25}$$

$$E_{\alpha+\beta}\zeta_3 = \sqrt{2}\zeta_6 \tag{26}$$

all others being zero.

The diagonal operators have to be calculated after the non-diagonal in Cartan composition, as it is necessary to know the relevant basic vectors. In this case they are

$$\left.\begin{aligned} \zeta_1 &= \xi_1\eta_1, \\ \zeta_2 &= \frac{1}{\sqrt{2}}(\xi_2\eta_1+\xi_1\eta_2), \\ \zeta_3 &= \frac{1}{\sqrt{2}}(\xi_3\eta_1+\xi_1\eta_3), \\ \zeta_4 &= \xi_2\eta_2, \\ \zeta_5 &= \frac{1}{\sqrt{2}}(\xi_3\eta_2+\xi_2\eta_3), \\ \zeta_6 &= \xi_3\eta_3, \end{aligned}\right\} \tag{27}$$

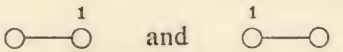
so the diagonal operators are, from equations (13), (27) and Theorem 12

$$\begin{aligned} F_\alpha &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ & & & 2 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & -2 \end{pmatrix} \\ F_\beta &= \begin{pmatrix} 2 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & -2 & & \\ & & & & -1 & \\ & & & & & 0 \end{pmatrix}. \end{aligned} \tag{28}$$

The choice of signs of the representation calculated here is only one of many possible. The signs of a Cartan composition are determined by the signs chosen in the original representations.

Example 2

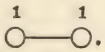
As a second example we take the combination of



and

and

whose greatest component is the representation



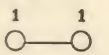
.

It is important in this example to choose the signs of the matrix elements of the two original representations correctly. Many combinations are possible. We shall take the representations

$$\left.\begin{aligned} E_\alpha{}^3{}_2 &= +1, & E_\alpha{}^2{}_1 &= +1, \\ E_\beta{}^2{}_1 &= +1, & E_\beta{}^3{}_2 &= +1, \\ E_{\alpha+\beta}{}^3{}_1 &= +1, & E_{\alpha+\beta}{}^3{}_1 &= -1. \end{aligned}\right\} \tag{29}$$

It will be seen that $N_{\alpha,\beta}$ is positive for one of these and negative for the other.

Applying Theorem 12 to the greatest vector of



using equations (29) we find

$$\left.\begin{aligned} E_\alpha\zeta_1 &= \xi_1\eta_2 = \zeta_2, \\ E_\beta\zeta_1 &= \xi_2\eta_1 = \zeta_3, \\ E_{\alpha+\beta}\zeta_1 &= \xi_3\eta_1-\xi_1\eta_3 = \sqrt{2}\zeta_4, \end{aligned}\right\} \tag{30}$$

say. Continuing

$$\left.\begin{aligned} E_\alpha\zeta_2 &= 0, \\ E_\beta\zeta_2 &= \xi_2\eta_2+\xi_1\eta_3 = \zeta_5', \\ E_{\alpha+\beta}\zeta_2 &= \xi_3\eta_2 = \zeta_6. \end{aligned}\right\} \tag{31}$$

It is now seen that the basic vector resulting from the second of equations (31) is not orthogonal to ζ_4 defined by the third of equations (30). It follows that ζ_5' is a linear combination of ζ_4 and ζ_5 where ζ_5 is a basic vector orthogonal to ζ_4, i.e.

$$\zeta_5' = a\zeta_4+b\zeta_5, \tag{32}$$

where

$$(\zeta_4,\zeta_5) = 0 \tag{33}$$

and

$$\zeta_5 = a_1\xi_1\eta_3+a_2\xi_2\eta_2+a_3\xi_3\eta_1. \tag{34}$$

It follows from equation (33) that

$$a_1 = a_3 \tag{35}$$

and from normalising ζ_5

$$a_1^2 + a_2^2 + a_3^2 = 2a_1^2 + a_2^2 = 1. \tag{36}$$

From equation (31) and equation (32) we have

$$\left.\begin{aligned} \frac{a}{\sqrt{2}} + a_1 b &= 0, \\ b a_2 &= 1, \\ b a_1 - \frac{a}{\sqrt{2}} &= 1, \end{aligned}\right\} \tag{37}$$

giving

$$b a_1 = -\frac{a}{\sqrt{2}} = 1 + \frac{a}{\sqrt{2}}, \tag{38}$$

or

$$a = -\frac{1}{\sqrt{2}} \tag{39}$$

and

$$a_1 = \frac{1}{2b}. \tag{40}$$

Since

$$a_2 = \frac{1}{b} \tag{41}$$

equation (36) gives

$$b^2 = \tfrac{3}{2}, \tag{42}$$

so

$$\zeta_5 = \frac{1}{\sqrt{6}}(\xi_1\eta_3 + 2\xi_2\eta_2 + \xi_3\eta_1) \tag{43}$$

and

$$\zeta_5' = \sqrt{\tfrac{3}{2}}\zeta_5 - \sqrt{\tfrac{1}{2}}\zeta_4. \tag{44}$$

Proceeding with the calculation of matrix elements and basic vectors we have

$$\left.\begin{aligned} E_\alpha\zeta_3 &= \xi_3\eta_1 + \xi_2\eta_2 = \sqrt{\tfrac{3}{2}}\zeta_5 + \sqrt{\tfrac{1}{2}}\zeta_4, \\ E_\beta\zeta_3 &= 0, \\ E_{\alpha+\beta}\zeta_3 &= -\xi_2\eta_3 = -\zeta_7, \end{aligned}\right\} \tag{45}$$

$$\left.\begin{aligned} E_\alpha\zeta_4 &= \sqrt{\tfrac{1}{2}}\zeta_6, \\ E_\beta\zeta_4 &= -\sqrt{\tfrac{1}{2}}\zeta_7, \\ E_{\alpha+\beta}\zeta_4 &= -\sqrt{2}\zeta_8, \end{aligned}\right\} \tag{46}$$

$$\left.\begin{aligned} E_\alpha\zeta_5 &= \sqrt{\tfrac{3}{2}}\zeta_6, \\ E_\beta\zeta_5 &= \sqrt{\tfrac{3}{2}}\zeta_7, \\ E_{\alpha+\beta}\zeta_5 &= 0, \end{aligned}\right\} \tag{47}$$

$$\left.\begin{aligned} E_\alpha\zeta_6 &= 0, \\ E_\beta\zeta_6 &= \zeta_8, \\ E_{\alpha+\beta}\zeta_6 &= 0, \end{aligned}\right\} \tag{48}$$

$$\left.\begin{aligned} E_\alpha\zeta_7 &= \zeta_8, \\ E_\beta\zeta_7 &= 0, \\ E_{\alpha+\beta}\zeta_7 &= 0. \end{aligned}\right\} \tag{49}$$

Comparison with Example 3 of Chapter 4 shows that this is the same representation. The differences are due to the arbitrary choice of a basis in the degenerate space, or to the arbitrary labelling of the basic vectors.

5.3 Analysis of a Kronecker product

The first step is to discover which irreducible representations occur in the direct sum. The dimension of the Kronecker product (being the product of the dimensions of the original representations) is equal to the sum of the dimensions of the irreducible representations. The greatest weight of the greatest component is known (the sum of the greatest weights of the original representations) so its dimension can be found. The greatest weight and therefore the dimension of the second greatest component follows from Theorem 13 below. For the rest, the Kronecker product is analysed by means of weight diagrams following Theorem 12. To do a complete analysis it would be necessary to know the weight diagrams of all the representations which might occur in the product. In practice, guess work and intuition can replace a lot of this work as will be seen in the examples.

Theorem 13

The power of the weight to which a basic vector of a Kronecker product corresponds is equal to the sum of the powers of the weights to which the component basic vectors correspond. That is, if

$$\zeta_i = \xi_r \times \eta_s \tag{50}$$

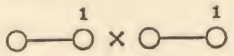
then the power of the weight to which ζ_t corresponds is

$$\delta(\lambda_r)+\delta(\lambda'_s), \tag{51}$$

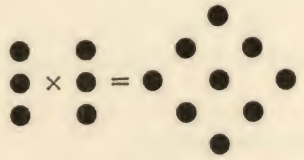
where $\delta(\lambda_r)$ is the power of the weight λ_r to which the basic vector ξ_r corresponds and $\delta(\lambda'_s)$ is that of the weight λ'_s to which η_s corresponds.

The total weight diagram of a Kronecker product can therefore be drawn directly from knowledge of the weight diagrams of the original representations. For example:

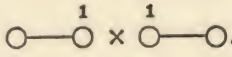
The weight diagram representing the Kronecker product



is

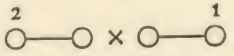


It is identical to the weight diagram representing the Kronecker product

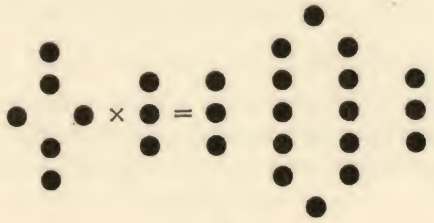


It is in the decomposition of the right-hand side that the difference appears.

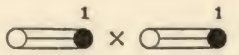
Similarly the weight diagram for



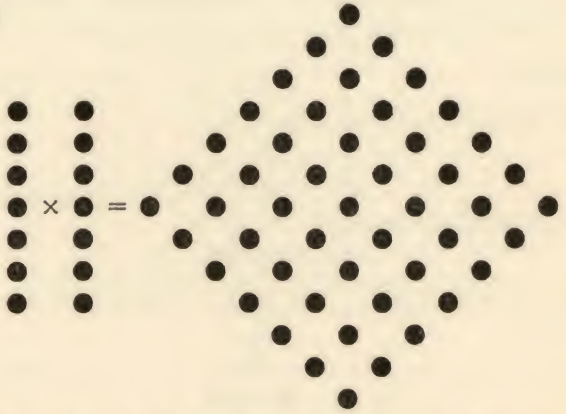
is



and that of



is the array



The greatest weight of the greatest component is known, so its weight diagram can be separated off. The greatest weight of the second greatest component follows from Theorem 14.

Theorem 14 **[11]**

The greatest weight of the second greatest irreducible representation of a Kronecker product is given by

$$\lambda'' = \lambda+\lambda'-(\alpha_1+\alpha_2+\ldots+\alpha_n), \tag{52}$$

where λ and λ' are the greatest weights of the original representations and $\alpha_1, \alpha_2, \ldots, \alpha_n$ is the 'minimal chain' joining λ and λ'. This minimal chain is the series of simple roots

$$\alpha_1, \alpha_2, \ldots, \alpha_n$$

which are such that

$$\left.\begin{array}{ll} (\lambda, \alpha_1) \neq 0, & \\ (\alpha_i, \alpha_{i+1}) \neq 0, & i = 1, 2, \ldots, n-1, \\ (\alpha_n, \lambda') \neq 0, & \end{array}\right\} \tag{53}$$

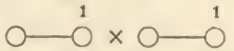
while

$$\left.\begin{array}{c} (\lambda, \alpha_2) = (\lambda, \alpha_3) = \ldots = (\lambda, \alpha_n) = 0 \\ (\lambda', \alpha_1) = (\lambda', \alpha_2) = \ldots = (\lambda', \alpha_{n-1}) = 0 \\ (\alpha_i, \alpha_j) = 0, \quad j \neq i \pm 1, i. \end{array}\right\} \tag{54}$$

The weight diagram of the representation corresponding to this greatest weight can then be separated off from the total weight diagram for the Kronecker product. The remaining weight diagram can usually be analysed quite simply although as the dimension of the Kronecker product increases, alternative analyses occur.

Example 3

Consider the Kronecker product

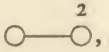


each representation having

$$\tfrac{1}{3}\alpha+\tfrac{2}{3}\beta$$

for its greatest weight. The weight diagram of the Kronecker product is

We know the greatest component is



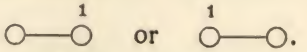
with greatest weight

$$\tfrac{2}{3}\alpha+\tfrac{4}{3}\beta.$$

This is a six-dimensional representation. Since the power of its greatest weight is two, it must have a weight diagram

taking into account the spindle shape described in 3.4. When this is subtracted from the total weight diagram it leaves

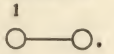
which is the weight diagram for



Applying Theorem 14 we find

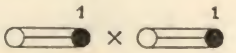
$$\lambda' = \tfrac{2}{3}\alpha+\tfrac{4}{3}\beta-\beta = \tfrac{2}{3}\alpha+\tfrac{1}{3}\beta \qquad (55)$$

and the second greatest component is



Example 4

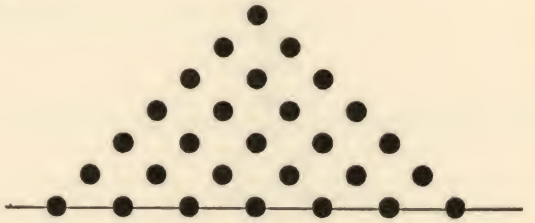
For the Kronecker product



where each has the greatest weight

$$\alpha+2\beta$$

we get the array

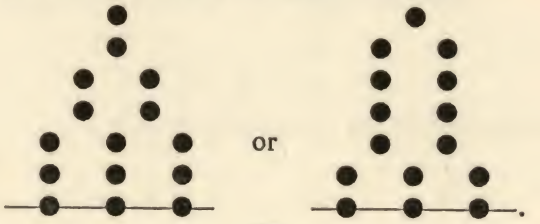


(the lower half is the reflection of the upper half in the zero line). The greatest component will be

with greatest weight

$$2\alpha+4\beta$$

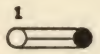
and of dimension 27. Its weight diagram will therefore start on line six with one spot, and will have 27 spots altogether. It will therefore have an odd number on the zero line, i.e.



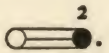
From Theorem 14 the greatest weight of the second greatest component will be

$$2\alpha+4\beta-\beta=2\alpha+3\beta, \tag{56}$$

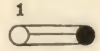
i.e.



of dimension fourteen. Since the power of its greatest weight is five it follows that the first alternative weight diagram above must be the one which corresponds to



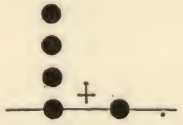
The weight diagram for



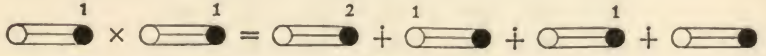
must be

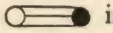
in order to have a greatest weight of power five and be of dimension fourteen. Subtracting these two from the original array leaves

which can only be the direct sum



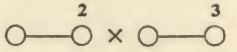
The Kronecker product can therefore be written



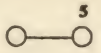
where  implies a zero greatest weight and a one-dimensional representation.

Example 5

For a third example we shall briefly examine a Kronecker product whose second greatest component is of higher dimension than the greatest component. Consider the product:



the representations being of dimension six and ten respectively. The greatest component is



of dimension 21 and greatest weight

$$\tfrac{5}{3}\alpha+\tfrac{10}{3}\beta.$$

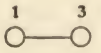
The greatest weight of the second greatest component is therefore

$$\tfrac{5}{3}\alpha+\tfrac{7}{3}\beta$$

with

$$\left.\begin{aligned}\lambda_\alpha &= 1,\\ \lambda_\beta &= 3,\end{aligned}\right\} \tag{57}$$

i.e.



which has dimension 24. The rest of the direct product can be analysed by means of weight diagrams making use of Appendix 1.

5.4 Calculation of a component of a Kronecker product

In order to find any component irreducible representation of a Kronecker product explicitly it is only necessary to know the basic vector to which its greatest weight corresponds, and then to work through the procedure given for the Cartan composition. The basic vector to which the greatest weight of the second greatest component corresponds is given by Theorem 15.

Theorem 15 [11]

The basic vector to which the greatest weight λ'' of the second greatest representation corresponds is given by

$$\zeta_1' \propto \sum_{s=0}^{k} a_s \xi_s \eta_{k-s} \tag{58}$$

where

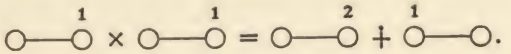
$$\left.\begin{aligned} \xi_0 &= \xi_\lambda, \\ \xi_s &= \phi(E_{-\alpha_s})\phi(E_{-\alpha_{s-1}})\ldots\phi(E_{-\alpha_1})\xi_\lambda, \\ &\qquad s = 1,2,\ldots,k, \\ \eta_0 &= \eta_{\lambda'}, \\ \eta_{k-s} &= \phi'(E_{-\alpha_{s+1}})\phi'(E_{-\alpha_{s+2}})\ldots\phi'(E_{-\alpha_k})\eta_{\lambda'}, \\ &\qquad s = 0,1,\ldots,k-1, \\ a_0 &= (\lambda,\alpha_1), \\ a_s &= (-1)^{s+1}(\alpha_s,\alpha_{s+1}), \quad s = 1,2,\ldots,k-1, \\ a_k &= (-1)^k(\alpha_k,\lambda'), \end{aligned}\right\} \tag{59}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k$ is the minimal chain joining λ and λ'. The proportional sign in equation (58) is a result of the necessity that ζ_1' be normalised.

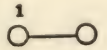
For any other component the basic vector corresponding to the greatest weight must be found by working through all the representations. The power of the greatest weight will be known so the basic vectors corresponding to weights of the same power occurring in other representations are found. The desired basic vector will be orthogonal to all these (and will correspond to a weight of the same power). Examples will clarify this.

Example 6

As an example of the use of Theorem 15, consider the Kronecker product



The greatest weight of the second greatest component of this product, i.e.



corresponds to the basic vector

$$\zeta_1' \propto \sum_{s=0}^{1} a_s \xi_s \eta_{k-s}, \tag{60}$$

where

$$\left.\begin{aligned} \xi_0 &= \xi_{\lambda_1}, & \eta_0 &= \eta_{\lambda_1}, \\ \xi_1 &= E_\beta \xi_{\lambda_1} = \xi_{\lambda_2}, & \eta_1 &= E_\beta \eta_{\lambda_1} = \eta_{\lambda_2}, \\ a_0 &= (\tfrac{1}{3}\alpha + \tfrac{2}{3}\beta, \beta) = 1, & a_1 &= (-1)(\tfrac{1}{3}\alpha + \tfrac{2}{3}\beta, \beta) = -1, \end{aligned}\right\} \tag{61}$$

so

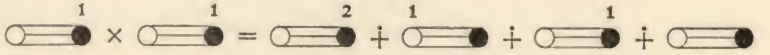
$$\zeta_1' \propto \xi_{\lambda_1}\eta_{\lambda_2} - \xi_{\lambda_2}\eta_{\lambda_1}. \tag{62}$$

Comparison with Example 1 shows that this vector is orthogonal to the basic vector ζ_2 which corresponds to a weight of the same power.

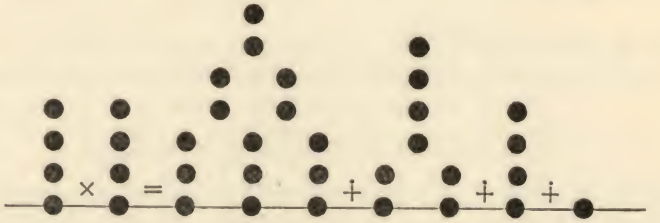
Example 7

We shall now find the greatest weight of the representation

in the product



of Example 4. The weight diagram of this product is



It is necessary to find all the basic vectors corresponding to weights of power four. These will be ζ_5 and ζ_6 of the greatest component, ζ_3' of the second greatest and ζ_1'' of the component we are interested in.

ζ_5 and ζ_6 are found direct from the greatest vector

$$\zeta_1 = \xi_1 \eta_1 \tag{63}$$

using the representation

as calculated in Chapter 4, Example 2, but with a consistent set of signs:

$$\left.\begin{array}{lll} E_\alpha{}^3{}_2 = +1, & E_\beta{}^7{}_6 = +\sqrt{\tfrac{1}{3}}, & E_{\alpha+\beta}{}^7{}_5 = -\sqrt{\tfrac{1}{3}} \\ E_\alpha{}^6{}_5 = +1, & E_\beta{}^5{}_4 = +\sqrt{\tfrac{2}{3}}, & E_{\alpha+\beta}{}^6{}_4 = +\sqrt{\tfrac{2}{3}} \\ & E_\beta{}^4{}_3 = +\sqrt{\tfrac{2}{3}}, & E_{\alpha+\beta}{}^4{}_2 = -\sqrt{\tfrac{2}{3}} \\ & E_\beta{}^2{}_1 = +\sqrt{\tfrac{1}{3}}, & E_{\alpha+\beta}{}^3{}_1 = +\sqrt{\tfrac{1}{3}} \\ E_{\alpha+2\beta}{}^7{}_4 = -\sqrt{\tfrac{2}{3}}, & & \\ E_{\alpha+2\beta}{}^6{}_3 = +\sqrt{\tfrac{1}{3}}, & E_{\alpha+3\beta}{}^7{}_3 = -1, & E_{2\alpha+3\beta}{}^7{}_2 = -1 \\ E_{\alpha+2\beta}{}^5{}_2 = +\sqrt{\tfrac{1}{3}}, & E_{\alpha+3\beta}{}^5{}_1 = +1, & E_{2\alpha+3\beta}{}^6{}_1 = -1. \\ E_{\alpha+2\beta}{}^4{}_1 = -\sqrt{\tfrac{2}{3}}, & & \end{array}\right\} \quad (64)$$

This set of signs depends on the choice

$$\left.\begin{array}{r} N_{\alpha,\beta} = +1, \\ N_{\alpha+\beta,\beta} = +\dfrac{2}{\sqrt{3}}, \\ N_{\alpha+2\beta,\beta} = +1, \\ N_{\alpha+3\beta,\alpha} = +1, \\ N_{\alpha+2\beta,\alpha+\beta} = -1. \end{array}\right\} \quad (65)$$

The calculation goes as follows:

$$\left.\begin{array}{r} E_\alpha\zeta_1 = 0, \\ E_\beta\zeta_1 = \sqrt{\tfrac{1}{3}}(\xi_1\eta_2+\xi_2\eta_1) = \sqrt{\tfrac{2}{3}}\zeta_2, \\ E_{\alpha+\beta}\zeta_1 = \sqrt{\tfrac{1}{3}}(\xi_1\eta_3+\xi_3\eta_1) = \sqrt{\tfrac{2}{3}}\zeta_3, \\ E_{\alpha+2\beta}\zeta_1 = -\sqrt{\tfrac{2}{3}}(\xi_1\eta_4+\xi_4\eta_1) = -\sqrt{\tfrac{4}{3}}\zeta_5, \end{array}\right\} \quad (66)$$

where ζ_5 is labelled such (leaving out ζ_4) because the weight to which it corresponds is three, and a second basic vector (besides ζ_3) corresponding to a weight of power four is expected. Other operators applied to ζ_1 will produce weights with power less than four so are not of interest. Continuing we have

$$\left.\begin{array}{r} E_\alpha\zeta_2 = \zeta_3, \\ E_\beta\zeta_2 = \sqrt{\tfrac{2}{3}}\xi_2\eta_2 = \sqrt{\tfrac{2}{3}}\zeta_4, \\ E_{\alpha+\beta}\zeta_2 = \sqrt{\tfrac{1}{6}}(\xi_3\eta_2+\xi_2\eta_3), \\ -\sqrt{\tfrac{1}{3}}(\xi_1\eta_4+\xi_4\eta_1), \\ = \sqrt{\tfrac{1}{3}}\zeta_6-\sqrt{\tfrac{2}{3}}\zeta_5, \end{array}\right\} \quad (67)$$

where

$$\left.\begin{array}{l} \zeta_6 = \dfrac{1}{\sqrt{2}}(\xi_3\eta_2+\xi_2\eta_3), \\ \zeta_5 = \dfrac{1}{\sqrt{2}}(\xi_1\eta_4+\xi_4\eta_1). \end{array}\right\} \quad (68)$$

In order to find ζ_3' it is necessary to find ζ_1' which will be orthogonal to ζ_2, i.e.

$$\zeta_1' = \frac{1}{\sqrt{2}}(\xi_1\eta_2-\xi_2\eta_1), \quad (69)$$

then

$$\left.\begin{array}{r} E_\alpha\zeta_1' = \dfrac{1}{\sqrt{2}}(\xi_1\eta_3-\xi_3\eta_1) = \zeta_2', \\ E_\beta\zeta_1' = 0, \\ E_{\alpha+\beta}\zeta_1' = \sqrt{\tfrac{1}{6}}(\xi_3\eta_2-\xi_2\eta_3)-\sqrt{\tfrac{1}{3}}(\xi_1\eta_4-\xi_4\eta_1), \\ = \zeta_3'. \end{array}\right\} \quad (70)$$

Since ζ_1'' is orthogonal to ζ_5, ζ_6 and ζ_3',

$$\zeta_1'' = \sqrt{\tfrac{1}{3}}(\xi_3\eta_2-\xi_2\eta_3)+\sqrt{\tfrac{1}{6}}(\xi_1\eta_4-\xi_4\eta_1). \quad (71)$$

5.5 Summary of Chapter 5

1. If Φ is the Kronecker product of the representations ϕ and ψ, then the matrix representing the operator T_a in the representation Φ is given by

$$\Phi(T_a)\zeta_i = [\phi(T_a)\xi_r]\times\eta_s+\xi_r\times[\psi(T_a)\eta_s],$$

where $\phi(T_a)$ is the matrix representing the operator T_a in the representation ϕ with basic vectors ξ_t $(t = 1, 2, \ldots, n)$ and $\psi(T_a)$ is the matrix representing T_a in the representation ψ having basic vectors η_s $(s = 1, 2, \ldots, n)$.

2. A Kronecker product is in general a reducible representation being the direct sum of a number of irreducible representations. In order to find one component of the direct sum explicitly, one starts with the basic vector corresponding to its greatest weight and applies the above formula to it. The resulting vectors are taken to be also basic vectors of the representation. (This statement should be modified if the representation has a degenerate weight system.)

3. The basic vector corresponding to the greatest weight of the greatest component is the product of the basic vectors of the original representations which correspond to their greatest weights. The greatest weight of the second greatest irreducible representation is given by

$$\lambda'' = \lambda + \lambda' - (\alpha_1 + \alpha_2 + \ldots + \alpha_k),$$

where λ and λ' are the greatest weights of the original representations and $\alpha_1, \alpha_2, \ldots, \alpha_k$ is the minimal chain joining λ and λ'.

The basic vector to which the greatest weight λ'' of the second greatest component corresponds is given by

$$\zeta_1' \propto \sum_{s=0}^{k} a_s \xi_s \eta_{k-s},$$

where

$$\xi_0 = \xi_\lambda,$$

$$\xi_s = \phi(E_{-\alpha_s})\phi(E_{-\alpha_{s-1}}) \ldots \phi(E_{-\alpha_1})\xi_\lambda, \quad s = 1, 2, \ldots, k,$$

$$\eta_0 = \eta_{\lambda'},$$

$$\eta_{k-s} = \phi'(E_{-\alpha_{s+1}})\phi'(E_{-\alpha_{s+2}}) \ldots \phi'(E_{-\alpha_k})\eta_{\lambda'}, \quad s = 0, 1, \ldots, k-1,$$

$$a_0 = (\lambda, \alpha_1),$$

$$a_s = (-1)^{s+1}(\alpha_s, \alpha_{s+1}), \quad s = 1, 2, \ldots, k-1,$$

$$a_k = (-1)^k(\alpha_k, \lambda'),$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k$ is the minimal chain joining λ and λ'. The vector ζ_1' should be normalized.

Basic vectors corresponding to greatest weights of other components must be found by working through the representations.

CHAPTER 6

Calculation of Basic Representations by Alternation

6.1 Alternation

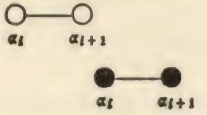
Alternation is the name given to the method of deriving basic representations from elementary representations.

Let α_r be a simple root of a simple Lie algebra, and ϕ_{α_r} be the representation corresponding to the Schouten diagram having all components of the greatest weight zero except that corresponding to α_r which is unity. If α_r is a terminal simple root, i.e. is connected to only one other simple root in the Schouten diagram (in other words is orthogonal to all other simple roots but one) then the representation is called elementary. If α_r is not a terminal simple root, the representation is called basic. Basic representations can be calculated by means of alternation from the elementary representation corresponding to the terminal vector on whose branch they occur.

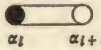
The branch of a terminal simple root is the series

$$B_\alpha = \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_m, \tag{1}$$

where each simple root α_i $(i = 0, 1, \ldots, m)$ is connected only to α_{i-1} and α_{i+1} the connections having one of the forms



or



$$i = 0, 1, \ldots, m.$$

The basic representation ϕ_{α_k} where α_k is the kth simple root on the branch corresponding to α_0 is the 'kth alternation' of the elementary representation ϕ_{α_0}. It can be calculated when ϕ_{α_0} is known, by means of Theorem 16.

Theorem 16 [11]

Suppose $\eta_j\ (j = 1, 2, \ldots, n)$ are the basic vectors of the elementary representation ϕ_{α_0}. Then

$$\zeta_s = \eta_{i_1}\eta_{i_2}\ldots\eta_{i_k} \tag{2}$$

are the basic vectors of ϕ_{α_k} where the $\eta_{i_1}, \eta_{i_2}, \ldots, \eta_{i_k}$ are a set of k different basic vectors of ϕ_{α_0} and

$$\eta_{j_1}\eta_{j_2}\ldots\eta_{j_k} = \varepsilon^{i_1,i_2,\ldots,i_k}_{j_1,j_2,\ldots,j_k}\eta_{i_1}\eta_{i_2}\ldots\eta_{i_k}, \tag{3}$$

where

$$\varepsilon^{i_1 i_2\ldots i_k}_{j_1 j_2\ldots j_k} = +1 \tag{4}$$

if the permutation $\begin{pmatrix} i_1, i_2, \ldots, i_k \\ j_1, j_2, \ldots, j_k \end{pmatrix}$ is even and

$$\varepsilon^{i_1 i_2\ldots i_k}_{j_1 j_2\ldots j_k} = -1 \tag{5}$$

if it is odd.

The representation ϕ_{α_k} is given by

$$\phi_{\alpha_k}\zeta_s = (\phi_{\alpha_0}\eta_{i_1})\eta_{i_2}\ldots\eta_{i_k} + \eta_{i_1}(\phi_{\alpha_0}\eta_{i_2})\ldots\eta_{i_k} + \\ + \ldots + \eta_{i_1}\eta_{i_2}\ldots(\phi_{\alpha_0}\eta_{i_k}). \tag{6}$$

It will be seen that the dimension of the kth alternation is

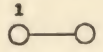
$${}^nC_k,$$

where n is the dimension of the elementary representation.

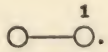
In general alternations give reducible representations and the greatest component must be separated in a similar way to that used for Cartan composition. In practice for the algebras A_n, B_n and D_n $(n \geqslant 5)$ alternations lead to irreducible representations. Only for the algebras C_n $(n \geqslant 3)$ and the exceptional algebras is the separation necessary.

Example 1

For a first example we take the simple case of calculating



from



Each of these representations is on the other's branch. It is a first alternation which is required. The calculation is of interest as it throws light on the question of signs in relation to this representation.

We start with the representation

$$\left.\begin{aligned} E_\alpha{}^3{}_2 &= +1, \\ E_\beta{}^2{}_1 &= +1, \\ E_{\alpha+\beta}{}^3{}_1 &= +1, \end{aligned}\right\} \tag{7}$$

$$F_\alpha = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad F_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{8}$$

with basic vectors ξ_{λ_1}, ξ_{λ_2} and ξ_{λ_3}. The first alternation of this representation will have for greatest basic vector

$$\xi_{\mu_1} = \xi_{\lambda_1}\xi_{\lambda_2}. \tag{9}$$

Applying Theorem 16

$$\left.\begin{aligned} E_\alpha\xi_{\mu_1} &= \xi_{\lambda_1}\xi_{\lambda_3} = \xi_{\mu_2}, \\ E_\beta\xi_{\mu_1} &= \xi_{\lambda_2}\xi_{\lambda_2} = 0, \\ E_{\alpha+\beta}\xi_{\mu_1} &= \xi_{\lambda_3}\xi_{\lambda_2} = \xi_{\mu_3}, \end{aligned}\right\} \tag{10}$$

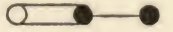
$$\left.\begin{aligned} E_\alpha\xi_{\mu_2} &= 0, \\ E_\beta\xi_{\mu_2} &= \xi_{\lambda_2}\xi_{\lambda_3} = -\xi_{\mu_3}, \\ E_{\alpha+\beta}\xi_{\mu_2} &= \xi_{\lambda_3}\xi_{\lambda_3} = 0, \end{aligned}\right\} \tag{11}$$

$$\left.\begin{aligned} E_\alpha\xi_{\mu_3} &= \xi_{\lambda_3}\xi_{\lambda_3} = 0, \\ E_\beta\xi_{\mu_3} &= 0, \\ E_{\alpha+\beta}\xi_{\mu_3} &= 0, \end{aligned}\right\} \tag{12}$$

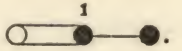
from which it is seen that, starting from a representation with $N_{\alpha,\beta} = +1$, however the basic vectors are chosen taking the first alternation leads to a representation with $N_{\alpha,\beta} = -1$.

Example 2

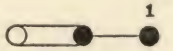
As a second example we examine the first alternation of



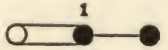
namely



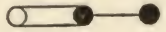
The representation



is of dimension six, so its first alternation will be of dimension fifteen, while

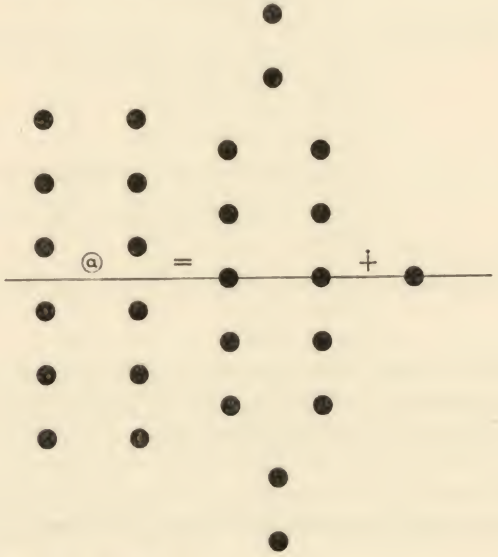


is of dimension fourteen. The alternation is therefore reducible into the greatest component as above, and the one-dimensional representation



having a zero weight vector.

Analysing by means of a weight diagram, using Theorem 12 which also holds for alternation and remembering that basic vectors of the final representation must contain different basic vectors of the original representation, we have



The greatest vector will be

$$\xi_{\lambda_1} = \xi_{\mu_1}\xi_{\mu_2} \tag{13}$$

and the representation is easily calculated by applying Theorem 16. The zero degeneracy (see Chapter 3) can be dealt with in the same way as it was for a Kronecker product, namely, by calling the first basic vector produced by the process and corresponding to a zero weight ξ_{λ_7}. When another is produced it is taken to be a linear combination of ξ_{λ_7} and ξ_{λ_8}, where

$$(\xi_{\lambda_7}, \xi_{\lambda_8}) = 0. \tag{14}$$

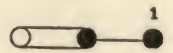
Alternatively, the basic vector corresponding to the one-dimensional representation can be found. It will be the linear combination $\xi_{\lambda_1'}$ of

$$\xi_{\mu_1}\xi_{\mu_6}, \quad \xi_{\mu_2}\xi_{\mu_5} \quad \text{and} \quad \xi_{\mu_3}\xi_{\mu_4}$$

which is such that

$$E_\delta \xi_{\lambda_1'} = 0 \tag{15}$$

for all $\delta \in \Sigma$. Using the representation of



which can be calculated directly from the weight system found by the methods of 3.3, and

$$\xi_{\lambda_1'} = a\xi_{\mu_1}\xi_{\mu_6} + b\xi_{\mu_2}\xi_{\mu_5} + c\xi_{\mu_3}\xi_{\mu_4} \tag{16}$$

we have

$$\left.\begin{aligned}
E_\alpha \xi_{\lambda_1'} &= 0, \\
E_\beta \xi_{\lambda_1'} &= b\xi_{\mu_3}\xi_{\mu_5} - c\xi_{\mu_3}\xi_{\mu_5} = 0, \\
E_\gamma \xi_{\lambda_1'} &= a\xi_{\mu_2}\xi_{\mu_6} + b\xi_{\mu_2}\xi_{\mu_6} = 0, \\
E_{\alpha+\beta} \xi_{\lambda_1'} &= b\xi_{\mu_4}\xi_{\mu_5} + c\xi_{\mu_5}\xi_{\mu_4} = 0, \\
E_{\beta+\gamma} \xi_{\lambda_1'} &= a\xi_{\mu_3}\xi_{\mu_6} + c\xi_{\mu_3}\xi_{\mu_6} = 0, \\
E_{\alpha+2\beta} \xi_{\lambda_1'} &= 0, \\
E_{\alpha+\beta+\gamma} \xi_{\lambda_1'} &= a\xi_{\mu_4}\xi_{\mu_6} - c\xi_{\mu_6}\xi_{\mu_4} = 0, \\
E_{\alpha+2\beta+\gamma} \xi_{\lambda_1'} &= a\xi_{\mu_5}\xi_{\mu_6} - b\xi_{\mu_6}\xi_{\mu_5} = 0, \\
E_{\alpha+2\beta+2\gamma} \xi_{\lambda_1'} &= 0,
\end{aligned}\right\} \tag{17}$$

i.e. $b = c = -a$, and from equation (16):

$$\xi_{\lambda_1'} = \frac{1}{\sqrt{3}}(\xi_{\mu_1}\xi_{\mu_6} - \xi_{\mu_2}\xi_{\mu_5} - \xi_{\mu_3}\xi_{\mu_4}). \tag{18}$$

The basic vectors ξ_{λ_7} and ξ_{λ_8} are then any two combinations of

$$\xi_{\mu_1}\xi_{\mu_6}, \quad \xi_{\mu_2}\xi_{\mu_5} \quad \text{and} \quad \xi_{\mu_3}\xi_{\mu_4}$$

orthogonal to $\xi_{\lambda_1'}$ and to each other. For example, one possibility is

$$\left.\begin{aligned}
\xi_{\lambda_7} &= \frac{1}{\sqrt{2}}(\xi_{\mu_2}\xi_{\mu_5} + \xi_{\mu_1}\xi_{\mu_6}) \\
\xi_{\lambda_8} &= \frac{1}{\sqrt{6}}(\xi_{\mu_1}\xi_{\mu_6} - \xi_{\mu_2}\xi_{\mu_5} + 2\xi_{\mu_3}\xi_{\mu_4}).
\end{aligned}\right\} \tag{19}$$

6.2 Summary of Chapter 6

If η_j $(j = 1, 2, \ldots, n)$ are the basic vectors of the elementary representation ϕ_{α_0} then

$$\zeta_s = \eta_{i_1}\eta_{i_2}\ldots\eta_{i_k}$$

are the basic vectors of the kth alternation ϕ_{α_k} where the $\eta_{i_1}, \eta_{i_2}, \ldots, \eta_{i_k}$ are a set of k different basic vectors of ϕ_{α_0} and

$$\eta_{j_1}\eta_{j_2}\ldots\eta_{j_k} = \varepsilon^{i_1 i_2 \ldots i_k}_{j_1 j_2 \ldots j_k}\eta_{i_1}\eta_{i_2}\ldots\eta_{i_k},$$

where

$$\varepsilon^{i_1 i_2 \ldots i_k}_{j_1 j_2 \ldots j_k} = +1$$

if the permutation is even and

$$\varepsilon^{i_1 i_2 \ldots i_k}_{j_1 j_2 \ldots j_k} = -1$$

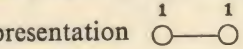
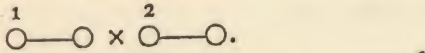
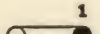
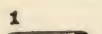
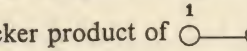
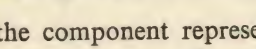
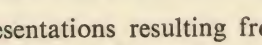
if it is odd.

The representation ϕ_{α_k} is given by

$$\phi_{\alpha_k}\zeta_s = (\phi_{\alpha_0}\eta_{i_1})\eta_{i_2}\ldots\eta_{i_k} + \eta_{i_1}(\phi_{\alpha_0}\eta_{i_2})\ldots\eta_{i_k} + \ldots + \eta_{i_1}\eta_{i_2}\ldots(\phi_{\alpha_0}\eta_{i_k}).$$

In general, alternations give reducible representations. The actual calculation and reduction follows the same lines as that for Kronecker products.

PROBLEMS 6

1. Find the basic vector corresponding to the greatest weight of the representation  in the Kronecker product  .
2. Calculate the representation  of B_2. From it, calculate the representation  by alternation.
3. Calculate the representation of the algebra A_2, first directly, and then by Cartan composition.
4. Find the component representations resulting from taking the Kronecker product of  with itself. (Use Appendix 1.)
5. Find the component representations resulting from taking the Kronecker product of  with  . (Use Appendix 1.)

APPENDIX 1

Tables concerning representations of A_2 or SU(3)

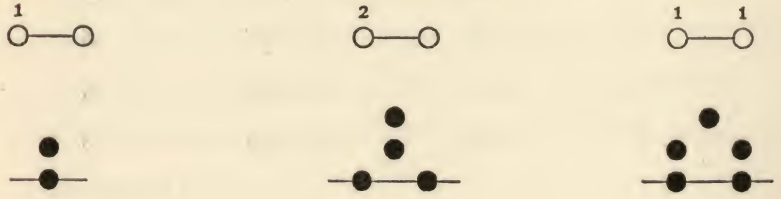
*Table A*1. Some properties of the representations of the algebra A_2 (or SU(3)) with lowest dimensions

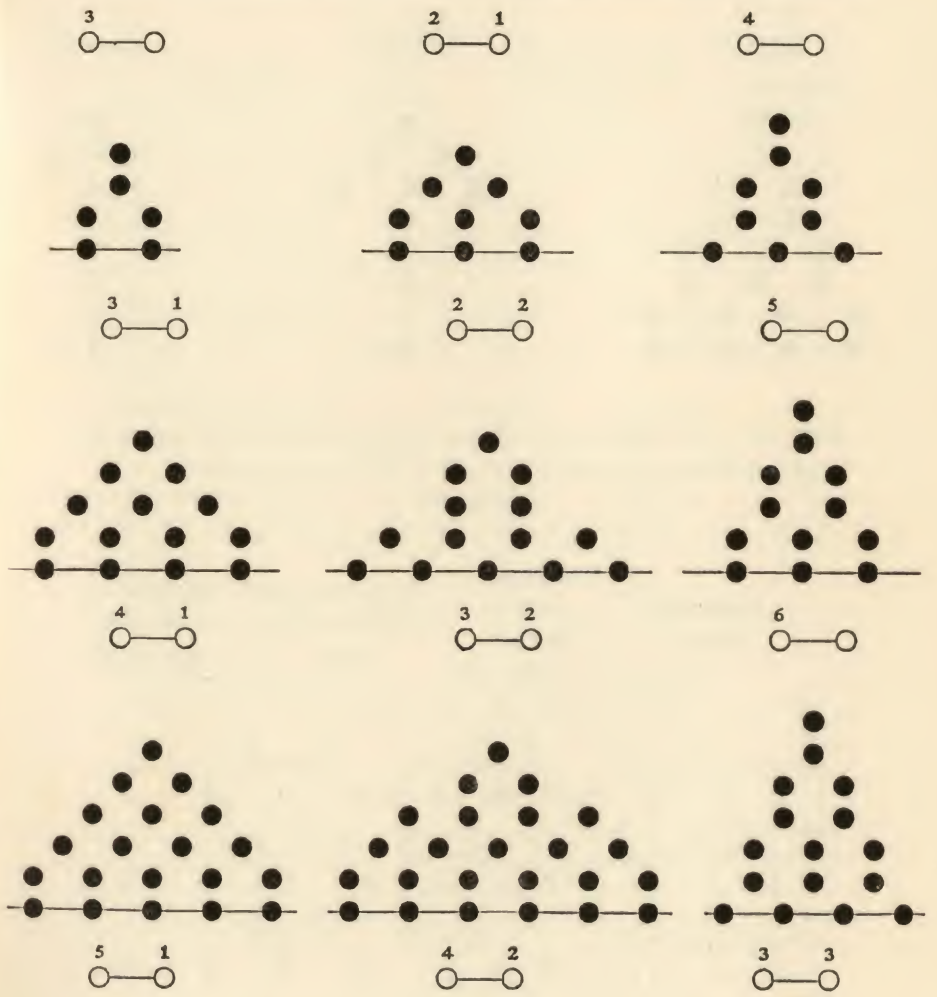
Representation	*Dimension*	*Greatest weight*	*Power of greatest weight*
	3	$\frac{2}{3}\alpha+\frac{1}{3}\beta$	1
	6	$\frac{4}{3}\alpha+\frac{2}{3}\beta$	2
	8	$\alpha+\beta$	2
	10	$2\alpha+\beta$	3
	15	$\frac{8}{3}\alpha+\frac{4}{3}\beta$	4
	15	$\frac{5}{3}\alpha+\frac{4}{3}\beta$	3
	21	$\frac{10}{3}\alpha+\frac{5}{3}\beta$	5
	24	$\frac{7}{3}\alpha+\frac{5}{3}\beta$	4
	27	$2\alpha+2\beta$	4
	28	$4\alpha+2\beta$	6
	35	$3\alpha+2\beta$	5
	36	$\frac{14}{3}\alpha+\frac{7}{3}\beta$	7
	42	$\frac{8}{3}\alpha+\frac{7}{3}\beta$	5
	45	$\frac{16}{3}\alpha+\frac{8}{3}\beta$	8
	48	$\frac{11}{3}\alpha+\frac{7}{3}\beta$	6

[continued overleaf

Representation	*Dimension*	*Greatest weight*	*Power of greatest weight*
4 2	55	$6\alpha+3\beta$	9
6 1	60	$\frac{10}{3}\alpha+\frac{8}{3}\beta$	6
3 3	63	$\frac{13}{3}\alpha+\frac{8}{3}\beta$	7
10	64	$3\alpha+3\beta$	6
11	66	$\frac{20}{3}\alpha+\frac{10}{3}\beta$	10
7 1	78	$\frac{22}{3}\alpha+\frac{11}{3}\beta$	11
5 2	80	$5\alpha+3\beta$	8
4 3	81	$4\alpha+3\beta$	7
12	90	$\frac{11}{3}\alpha+\frac{10}{3}\beta$	7
8 1	91	$8\alpha+4\beta$	12
13	99	$\frac{17}{3}\alpha+\frac{10}{3}\beta$	9
6 2	105	$\frac{26}{3}\alpha+\frac{13}{3}\beta$	13
5 3	105	$\frac{14}{3}\alpha+\frac{10}{3}\beta$	8
4 4	120	$\frac{13}{3}\alpha+\frac{11}{3}\beta$	8
	125	$4\alpha+4\beta$	8

Table A2. Weight diagrams of the representations of the algebra A_2 (or SU(3)) with lowest powers





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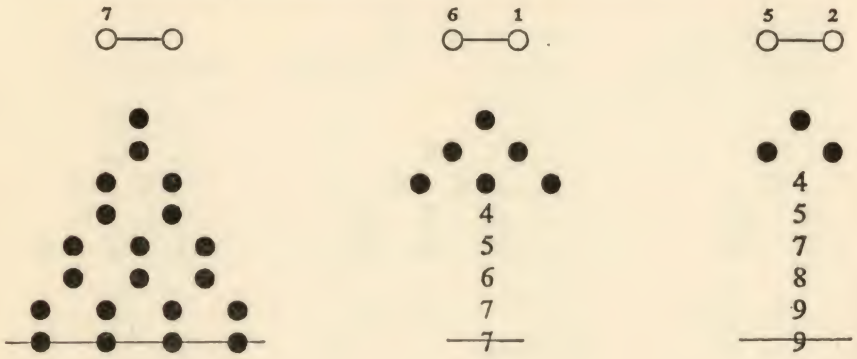

Table A3. The largest components of the Kronecker products of representations of the algebra A_2 (or SU(3)) having lowest dimension

Kronecker product	*Component with greatest power*	*Component with second greatest power*	*Number of remaining dimensions*
$3 \times 3 = 9$	$= 6$	$+ 3$	$+ 0$
$3 \times 3 = 9$	$= 8$	$+ 1$	$+ 0$
$3 \times 6 = 18$	$= 10$	$+ 8$	$+ 0$
$3 \times 6 = 18$	$= 15$	$+ 3$	$+ 0$
$3 \times 8 = 24$	$= 15$	$+ 6$	$+ 3$
$6 \times 6 = 36$	$= 15$	$+ 15$	$+ 6$

Kronecker product	*Component with greatest power*	*Component with second greatest power*	*Number of remaining dimensions*
$6 \times 6 = 36$	$= 27$	$+ 8$	$+ 1$
$6 \times 8 = 48$	$= 24$	$+ 15$	$+ 9$
$8 \times 8 = 64$	$= 27$	$+ 10$	$+ 27$
$3 \times 10 = 30$	$= 15$	$+ 15$	$+ 0$
$3 \times 10 = 30$	$= 24$	$+ 6$	$+ 0$
$6 \times 10 = 60$	$= 21$	$+ 24$	$+ 15$
$6 \times 10 = 60$	$= 42$	$+ 15$	$+ 3$

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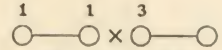
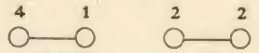
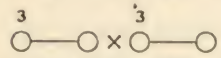
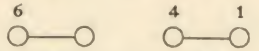
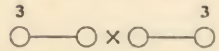
Appendix 1

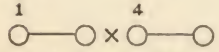


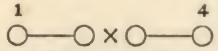


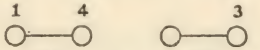


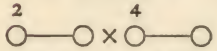


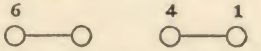


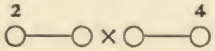


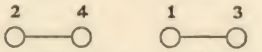


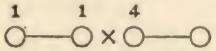


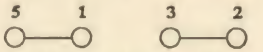




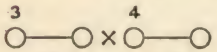


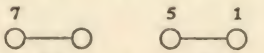






Kronecker product	*Component with greatest power*	*Component with second greatest power*	*Number of remaining dimensions*
$\overset{1}{\circ}\!-\!\overset{1}{\circ} \times \overset{3}{\circ}\!-\!\circ$		$\overset{4}{\circ}\!-\!\overset{1}{\circ}$ $\quad$ $\overset{2}{\circ}\!-\!\overset{2}{\circ}$	
8 × 10	= 80	= 35 + 27	+ 18
$\overset{3}{\circ}\!-\!\circ \times \overset{3}{\circ}\!-\!\circ$		$\overset{6}{\circ}\!-\!\circ$ $\quad$ $\overset{4}{\circ}\!-\!\overset{1}{\circ}$	
10 × 10	= 100	= 28 + 35	+ 37
$\overset{3}{\circ}\!-\!\circ \times \circ\!-\!\overset{3}{\circ}$		$\overset{3}{\circ}\!-\!\overset{3}{\circ}$ $\quad$ $\overset{2}{\circ}\!-\!\overset{2}{\circ}$	
10 × 10	= 100	= 64 + 27	+ 9
$\overset{1}{\circ}\!-\!\circ \times \overset{4}{\circ}\!-\!\circ$		$\overset{5}{\circ}\!-\!\circ$ $\quad$ $\overset{3}{\circ}\!-\!\overset{1}{\circ}$	
3 × 15	= 45	= 21 + 24	+ 0
$\overset{1}{\circ}\!-\!\circ \times \circ\!-\!\overset{4}{\circ}$		$\overset{1}{\circ}\!-\!\overset{4}{\circ}$ $\quad$ $\circ\!-\!\overset{3}{\circ}$	
3 × 15	= 45	= 35 + 10	+ 0
$\overset{2}{\circ}\!-\!\circ \times \overset{4}{\circ}\!-\!\circ$		$\overset{6}{\circ}\!-\!\circ$ $\quad$ $\overset{4}{\circ}\!-\!\overset{1}{\circ}$	
6 × 15	= 90	= 28 + 35	+ 27
$\overset{2}{\circ}\!-\!\circ \times \circ\!-\!\overset{4}{\circ}$		$\overset{2}{\circ}\!-\!\overset{4}{\circ}$ $\quad$ $\overset{1}{\circ}\!-\!\overset{3}{\circ}$	
6 × 15	= 90	= 60 + 24	+ 6
$\overset{1}{\circ}\!-\!\overset{1}{\circ} \times \overset{4}{\circ}\!-\!\circ$		$\overset{5}{\circ}\!-\!\overset{1}{\circ}$ $\quad$ $\overset{3}{\circ}\!-\!\overset{2}{\circ}$	
8 × 15	= 120	= 48 + 42	+ 30
$\overset{3}{\circ}\!-\!\circ \times \overset{4}{\circ}\!-\!\circ$		$\overset{7}{\circ}\!-\!\circ$ $\quad$ $\overset{5}{\circ}\!-\!\overset{1}{\circ}$	
10 × 15	= 150	= 36 + 48	+ 66





APPENDIX 2

Notation

Sets, groups, algebras, etc.	$\mathscr{R}, \mathscr{H}, \mathscr{N}$
Operators, members of sets, etc.	X, T, E_α, U_α
Structure constant	$f_a{}^c{}_b$
Vectors in idempotent, e.g. roots weights, etc.	$\alpha, \sigma, \mu, \lambda, \lambda_s$
Sets of vectors in idempotent	$\Sigma, \Pi, \Lambda, \Delta$
Real number coefficient	$a_\mu, a^\mu{}_\alpha$
Real number defining length	λ
Coefficient of commutation relation	$N_{\alpha,\beta}$
Dimension of representation	n, m, N
Vector in idempotent related to dimension of representation	g
Greatest weight of representation	λ
Component of greatest weight	$\lambda_\alpha = \dfrac{2(\lambda,\alpha)}{(\alpha,\alpha)}$
Basic vector, vector in representation space	$\xi, \xi_s, \xi_{\lambda_s}, \zeta, \eta$
Weight vector	λ_s
Power	$\delta(\mu), \delta(\lambda_s)$
Width of representation	s_k
Height of representation	T
Representation	ϕ, ψ, Φ
Branch	B_α

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